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Analytical-numerical solution of elliptical interface crack problem

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Abstract. The problem of elliptical interface crack, located between two bonded dissimilar elastic half spaces, is considered. To obtain a solution of the problem, the traction boundary pseudodifferential equations are used. An analytical-numerical method for solving these equations is proposed. Strain energy release rates along the crack contours are calculated for some examples.

Key words: Elliptical interface crack, traction boundary pseudodifferential equations, analytical-numerical method, strain energy release rate.

1. Formulation of the problem

Let a region G in the plane $x_3 = 0$ of an infinite elastic solid be occupied by a flat crack. Let the material of the upper half space ($x_3 > 0$) be defined by the Poisson's ratio ν_1 and the shear modulus μ_1, while the material of the lower half space ($x_3 < 0$) has the Poisson's ratio ν_2 and the shear modulus μ_2. Furthermore, let the loads applied to the crack surfaces be equal in magnitude and opposite in directions

$$\pm t(x) = \pm(t_1(x),\ t_2(x),\ t_3(x)),$$

where $x = (x_1, x_2)$.

The interface crack problem can be reduced to the following traction boundary pseudodifferential equations in the crack area (Willis, 1972; Goldstein, 1979)

$$P_G\, A(D)[u(x)] = t(x), \qquad x \in G \quad \text{and} \quad [u(x)] = 0, \qquad x \notin G, \tag{1}$$

where $[u(x)] = ([u_1(x)], [u_2(x)], [u_3(x)])$ are the crack opening displacements and $A(D)$ is a matrix pseudodifferential operator with the symbol

$$A(\xi) = \frac{|\xi|}{g^2 - d^2}\begin{pmatrix} g - e\,\eta_2^2 & e\,\eta_1\,\eta_2 & d i \eta_1 \\ e\,\eta_1\,\eta_2 & g - e\,\eta_1^2 & d i \eta_2 \\ -d i \eta_1 & -d\, i\, \eta_2 & g \end{pmatrix}. \tag{2}$$

In (1) and (2) the following notations are used

$$g = \frac{1-\nu_1}{\mu_1} + \frac{1-\nu_2}{\mu_2}, \qquad d = \frac{1}{2}\left(\frac{1-2\nu_1}{\mu_1} - \frac{1-2\nu_2}{\mu_2}\right),$$

$$e = \frac{g\,c + d^2}{g+c}, \qquad c = \frac{\nu_1}{\mu_1} + \frac{\nu_2}{\mu_2};$$

$$\xi = (\xi_1, \xi_2), \qquad |\xi| = \sqrt{\xi_1^2 + \xi_2^2}, \qquad \eta_j = \frac{\xi_j}{|\xi|}, \; j = 1, 2;$$

P_G is the restriction operator to the crack area G;

$$A(D)[u(x)] = F^{-1}(A(\xi)[\widetilde{u}(\xi)]),$$

where $[\widetilde{u}(\xi)] = \int_{R^2}[u(x)]\,\mathrm{e}^{i(x,\xi)}\,\mathrm{d}x$ is the Fourier transform, $(x,\xi) = x_1\xi_1 + x_2\xi_2$, and $F^{-1}(\widetilde{g}(\xi)) = 1/(2\pi)^2 \int_{R^2} \widetilde{g}(\xi)\,\mathrm{e}^{-i(x,\xi)}\,\mathrm{d}\xi$ is the inverse Fourier transform.

The solvability of the system, given by (1), in the Sobolev spaces $t(x) \in H_{-1/2}(G)$, $[u(x)] \in \overset{0}{H}_{1/2}(G)$ was proved by Goldstein and Shifrin (1981).

In what follows, our considerations will be restricted to the case when G is an elliptical region

$$G = \mathrm{Ela} = \left\{x\colon \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} \leqslant 1\right\}.$$

Without loss of generality, we assume that $a_2 \leqslant a_1$.

2. Solution: Analytical-numerical method

A number of papers have been devoted to the elliptical crack problem, dealing with the case of elliptical crack located in an infinite, homogeneous, elastic solid and subjected to the static polynomial loads (Vijayakumar and Atluri, 1981; Nishioka and Atluri, 1983; Borodachev, 1981; Martin, 1986a;b). An analytical method, recently developed for the same problem by Kaptsov and Shifrin (1991; 1995; 1996), was possible to apply for the solution of more complex problems; in particular, to the problem of elliptical crack subjected to an arbitrary time-harmonic loads (Kaptsov and Shifrin, 1991; Shifrin, 1996a;b; 1997). The aim of this paper is to extend the approach of Kaptsov and Shifrin to the solution of elliptical interface crack problem. In this section, the basic features of the proposed analytical-numerical method are presented.

Let us first denote the pseudodifferential operator with the symbol $|\xi|$ by Λ and the pseudodifferential operator with the symbol $|\xi|^{-1}$ by Λ^{-1}. Using these notations, equations (1) can be rewritten as

$$\frac{1}{g^2-d^2} P_G \left\{ g\,\Lambda[u_1(x)] + e\,\frac{\partial^2}{\partial x_2^2}\Lambda^{-1}[u_1(x)] - e\,\frac{\partial^2}{\partial x_1 \partial x_2}\Lambda^{-1}[u_2(x)] \right.$$
$$\left. - d\,\frac{\partial[u_3(x)]}{\partial x_1} \right\} = t_1(x),$$
$$\frac{1}{g^2-d^2} P_G \left\{ -e\,\frac{\partial^2}{\partial x_1 \partial x_2}\Lambda^{-1}[u_1(x)] + g\,\Lambda[u_2(x)] + e\,\frac{\partial^2}{\partial x_1^2}\Lambda[u_2(x)] \right. \tag{3}$$
$$\left. - d\,\frac{\partial[u_3(x)]}{\partial x_2} \right\} = t_2(x),$$
$$\frac{1}{g^2-d^2} P_G \left\{ d\,\frac{\partial[u_1(x)]}{\partial x_1} + d\,\frac{\partial[u_2(x)]}{\partial x_2} + g\,\Lambda[u_3(x)] \right\} = t_3(x).$$

Further, it is assumed, that the applied loads may be represented by the infinite power series

$$t_1(x) = \sum_{i,j=0}^{\infty} A_{ij}^{1}\, x_1^i\, x_2^j = \sum_{i,j=0}^{\infty} A_{ij}^{*1}\, y_1^i\, y_2^j,$$
$$t_2(x) = \sum_{i,j=0}^{\infty} A_{ij}^{*2}\, y_1^i\, y_2^j, \tag{4}$$
$$t_3(x) = \sum_{i,j=0}^{\infty} A_{ij}^{*3}\, y_1^i\, y_2^j,$$

where A_{ij}^{*1}, A_{ij}^{*2} and A_{ij}^{*3} are constants, and y_j are defined as

$$y_j = \frac{x_j}{a_j}, \qquad j = 1, 2.$$

Let us look for the solution of (3) in the following form

$$[u_1(x)] = \sum_{p,q=0}^{\infty} B_{pq}^{1}\, T_{pq}^{1/2}(x),$$
$$[u_2(x)] = \sum_{p,q=0}^{\infty} B_{pq}^{2}\, T_{pq}^{1/2}(x), \tag{5}$$
$$[u_3(x)] = \sum_{p,q=0}^{\infty} B_{pq}^{3}\, T_{pq}^{1/2}(x),$$

where B_{pq}^1, B_{pq}^2, B_{pq}^3 are constants, and functions $T_{pq}^{1/2}(x)$ are defined as

$$T_{pq}^{1/2}(x) = y_1^p\, y_2^q \sqrt{1 - y_1^2 - y_2^2}. \tag{6}$$

Let us introduce the following notations

$$
\begin{aligned}
&P_{\mathrm{Ela}}\ \Lambda\ T^{1/2}_{pq}(x) = U^{pq}(x),\\
&P_{\mathrm{Ela}}\ \frac{\partial^2}{\partial x_i \partial x_j}\ \Lambda^{-1}\, T^{1/2}_{pq}(x) = R^{pq}_{ij}(x), \quad i,\ j = 1, 2,\\
&\frac{\partial T^{1/2}_{pq}(x)}{\partial x_1} = L^{pq}_1(x),\\
&\frac{\partial T^{1/2}_{pq}(x)}{\partial x_2} = L^{pq}_2(x).
\end{aligned}
\tag{7}
$$

$U^{pq}(x)$ and $R^{pq}_{ij}(x)$ are polynomials of order $p+q$. Explicit expressions for these functions were obtained by Kaptsov and Shifrin (1995; 1996). For reader's convenience, they are provided in Appendix. Functions $L^{pq}_1(x)$ and $L^{pq}_2(x)$ can be expanded into infinite power series. Explicit expressions for these series are also given in Appendix.

Substituting (4) and (5) into equations (3), and using (7), we obtain

$$
\begin{aligned}
&\frac{1}{g^2-d^2}\left\{ g \sum_{p,q=0}^{\infty} B^1_{pq}\, U^{pq}(x) + e \sum_{p,q=0}^{\infty} B^1_{pq}\, R^{pq}_{22}(x) \right.\\
&\qquad \left. - e \sum_{p,q=0}^{\infty} B^2_{pq}\, R^{pq}_{12}(x) - d \sum_{p,q=0}^{\infty} B^3_{pq}\, L^{pq}_1(x) \right\} = \sum_{i,j=0}^{\infty} A^{*1}_{ij}\, y^i_1\, y^j_2,\\
&\frac{1}{g^2-d^2}\left\{ -e \sum_{p,q=0}^{\infty} B^1_{pq}\, R^{pq}_{12}(x) + g \sum_{p,q=0}^{\infty} B^2_{pq}\, U^{pq}(x) \right.\\
&\qquad \left. + e \sum_{p,q=0}^{\infty} B^2_{pq}\, R^{pq}_{11}(x) - d \sum_{p,q=0}^{\infty} B^3_{pq}\, L^{pq}_2(x) \right\} = \sum_{i,j=0}^{\infty} A^{*2}_{ij}\, y^i_1\, y^j_2,\\
&\frac{1}{g^2-d^2}\left\{ d \sum_{p,q=0}^{\infty} B^1_{pq}\, L^{pq}_1(x) + d \sum_{p,q=0}^{\infty} B^2_{pq}\, L^{pq}_2(x) \right.\\
&\qquad \left. + g \sum_{p,q=0}^{\infty} B^3_{pq}\, U^{pq}(x) \right\} = \sum_{i,j=0}^{\infty} A^{*3}_{ij}\, y^i_1\, y^j_2.
\end{aligned}
\tag{8}
$$

Equations (8) represent equalities of infinite power series. By equating the coefficients of the same power on the left hand side with the coefficients on the right hand side, we obtain an infinite system of linear algebraic equations, where constants B^1_{pq}, B^2_{pq} and B^3_{pq} are unknowns.

For solving (8) numerically, finite systems of linear equations, with $p+q \leqslant N$ and $i+j \leqslant N$, are considered.

3. Calculation of the strain energy release rate

In this section, the procedure to determine the strain energy release rate is presented. Let us assume, that the infinite system of (8) was solved (for a particular loading conditions) and, consequently, the values of constants B^1_{pq}, B^2_{pq} and B^3_{pq} are known. The aim is, to express the strain energy release rate J as a function of these constants.

Elliptical crack contour may be defined as

$$x_1 = a_1 \cos\theta \quad \text{and} \quad x_2 = a_2 \sin\theta. \tag{9}$$

At a point $(a_1 \cos\theta, a_2 \sin\theta)$, projections of a vector $([u_1(x)], [u_2(x)])$ on the directions $([u_n(x)])$ and $([u_\tau(x)])$ (directions normal and tangential to the crack contour, respectively) may be expressed as

$$\begin{aligned} [u_n(x)] &= \frac{[u_1(x)]a_2 \cos\theta + [u_2(x)]a_1 \sin\theta}{(a_1^2 \sin^2\theta + a_2^2 \cos^2\theta)^{1/2}}, \\ [u_\tau(x)] &= \frac{-[u_1(x)]a_1 \sin\theta + [u_2(x)]a_2 \cos\theta}{(a_1^2 \sin^2\theta + a_2^2 \cos^2\theta)^{1/2}}. \end{aligned} \tag{10}$$

It is known (Rice, 1988; Wang, Shin and Suo, 1992), that the crack opening displacements have the following asymptotics near the crack contour

$$[u_3(x)] + i\,[u_n(x)] \approx \frac{g}{(1 + 2i\,\varepsilon)\cosh(\pi\varepsilon)} K\, s^{i\varepsilon} \sqrt{\frac{2s}{\pi}}, \tag{11}$$

$$[u_\tau(x)] \approx \frac{2K_{\mathrm{III}}}{\mu_*} \sqrt{\frac{2s}{\pi}}, \tag{12}$$

where s is a distance between a point x and the crack contour, $K = K_{\mathrm{I}} + i\,K_{\mathrm{II}}$ is the complex stress intensity factor, while ε and μ_* are defined as

$$\varepsilon = \frac{1}{2\pi} \ln \frac{\mu_2 \kappa_1 + \mu_1}{\mu_1 \kappa_2 + \mu_2}, \qquad \kappa_j = 3 - 4\nu_j, \qquad j = 1, 2,$$
$$\mu_* = \frac{2\mu_1 \mu_2}{\mu_1 + \mu_2}.$$

It is convenient to introduce a function

$$\varphi_a^{1/2}(x) = \sqrt{1 - y_1^2 - y_2^2},$$

such that $T^{1/2}_{pq}(x) = y_1^p\, y_2^q\, \varphi_a^{1/2}(x)$. Asymptotics of the function $\varphi_a^{1/2}(x)$ in the vicinity of the crack contour is the following

$$\varphi_a^{1/2}(x) \approx s^{1/2} \sqrt{\frac{2}{a_1 a_2}} \left(a_1^2 \sin^2\theta + a_2^2 \cos^2\theta\right)^{1/4}. \tag{13}$$

The crack opening displacements (5) and (10) may be written now as

$$
\begin{aligned}
[u_j(x)] &= [u_j^0(x)]\varphi_a^{1/2}(x), \quad j = 1, 2, 3, \\
[u_n(x)] &= [u_n^0(x)]\varphi_a^{1/2}(x), \\
[u_\tau(x)] &= [u_\tau^0(x)]\varphi_a^{1/2}(x),
\end{aligned} \tag{14}
$$

where $[u_j^0(x)]$, $[u_n^0(x)]$ and $[u_\tau^0(x)]$ are infinite power series.

From (10), (12) and (13) it follows, that K_{III} at a point x, with x belonging to the crack contour (i.e. at $s = 0$), can be expressed as

$$
K_{\mathrm{III}} = \frac{\mu_*}{2}\sqrt{\frac{\pi}{a_1 a_2}}\left\{\frac{-[u_1^0(x)]a_1 \sin\theta + [u_2^0(x)]a_2\cos\theta}{(a_1^2\sin^2\theta + a_2^2\cos^2\theta)^{1/4}}\right\}. \tag{15}
$$

It also follows from (5), (13) and (14), that for point x of the crack contour $[u_j^0(x)]$, $j = 1, 2$, can be expressed as

$$
\begin{aligned}
[u_1^0(x)] &= \sum_{p,q=0}^{\infty} B_{pq}^1 \cos^p\theta \, \sin^q\theta, \\
[u_2^0(x)] &= \sum_{p,q=0}^{\infty} B_{pq}^2 \cos^p\theta \, \sin^q\theta.
\end{aligned} \tag{16}
$$

By inserting (16) into (15), the expression for K_{III} may be given as follows

$$
\begin{aligned}
K_{\mathrm{III}} &= \frac{\mu_*}{2}\sqrt{\frac{\pi}{a_1 a_2}}\frac{1}{(a_1^2\sin^2\theta + a_2^2\cos^2\theta)^{1/4}} \\
&\quad \times \sum_{p,q=0}^{\infty}\left\{B_{pq}^2\, a_2\cos^{p+1}\theta\,\sin^q\theta - B_{pq}^1\, a_1\cos^p\theta\,\sin^{q+1}\theta\right\}.
\end{aligned} \tag{17}
$$

Using (11), (13) and (14), the following relation, valid in the vicinity of the crack contour, can be obtained

$$
\begin{aligned}
&([u_3^0(x)] + i[u_n^0(x)])\sqrt{\frac{\pi}{a_1 a_2}}(a_1^2\sin^2\theta + a_2^2\cos^2\theta)^{1/4} \\
&= \frac{g\{[K_{\mathrm{I}} + 2\varepsilon K_{\mathrm{II}}] + i[K_{\mathrm{II}} - 2\varepsilon K_{\mathrm{I}}]\}}{(1 + 4\varepsilon^2)\cosh(\pi\varepsilon)}[\cos(\varepsilon\ln s) + i\sin(\varepsilon\ln s)].
\end{aligned} \tag{18}
$$

Considering the equality between the moduli of the left hand side and the right hand side of (18), we may obtain the following expression

$$
\begin{aligned}
K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2 &= \frac{(1 + 4\varepsilon^2)\cosh^2(\pi\varepsilon)}{g^2}\frac{\pi}{a_1 a_2}(a_1^2\sin^2\theta + a_2^2\cos^2\theta)^{1/2} \\
&\quad \times\{[u_3^0(x)]^2 + [u_n^0(x)]^2\},
\end{aligned} \tag{19}
$$

Table 1. Dependence of $J_* = J\mu_1/\sigma^2 a$ on number of equations N. Penny-shaped crack with radius a; $\nu_1 = \nu_2 = 0.3$; $\mu_1 : \mu_2 = 1 : 4$.

N	0	1	2	3	4	5
J_*	0.265579	0.275671	0.269958	0.274720	0.271015	0.274313
N	6	7	8	9	10	11
J_*	0.271510	0.274080	0.271804	0.273927	0.272000	0.273817
N	12	13	14	15	16	17
J_*	0.272141	0.273735	0.272248	0.273670	0.272332	0.273618
N	18	19	20	21		
J_*	0.272400	0.273575	0.272456	0.273539		

where x is a point on the crack contour. With (5), (13) and (14), we can express $[u_3^0(x)]^2$ as

$$[u_3^0(x)]^2 = \sum_{p,q,m,n=0}^{\infty} B_{pq}^3\, B_{mn}^3 \cos^{p+m}\theta \sin^{q+n}\theta,$$

while the expression for $\left[u_n^0(x)\right]^2$ can be obtained through (10), (14) and (16).

Finally, by using (17) and (19), it is possible to calculate the strain energy release rate, J, as (Salganik, 1963; Wang, Shih and Suo, 1992)

$$J = \frac{g}{4\cosh^2(\pi\varepsilon)}(K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2) + \frac{1}{2\mu^*} K_{\mathrm{III}}^2. \tag{20}$$

4. Examples

The procedure described in Sections 2 and 3 was transformed into a computer code. In this section, the results for the normal uniform loading conditions

$$t(x) = (0, 0, \sigma)$$

are presented as an example.

As already mentioned in Section 2, the infinite system of (8) needs to be truncated in order to be solved numerically. Accordingly, finite systems of equations, with $p + q \leqslant N$ and $i + j \leqslant N$, were used to calculate the strain energy release rates presented below. To illustrate stabilization of the results with the growth of N, dependence of $J_* = J\mu_1/\sigma^2 a$ on N (for a particular case of penny-shaped crack of radius a and materials with the following characteristics: Poisson's ratios are $\nu_1 = \nu_2 = 0.3$ and shear moduli ratio is $\mu_1 : \mu_2 = 1 : 4$) is given in Table 1. Results in Table 1 indicate, that the connection between the normal and the shear crack opening displacements in (1) is weak. It is therefore possible to obtain reasonable results even when the infinite system of linear (8) is replaced by the finite system of equations of the low order. At the same time, the convergence of the results to the limiting value (when increasing N) is slow.

The results of the calculations, performed by the authors for nonhomogenous solids with various material constants and elliptical cracks with various semi-axes ratios show, that type of the results convergence for those cases is very similar to one presented in Table 1.

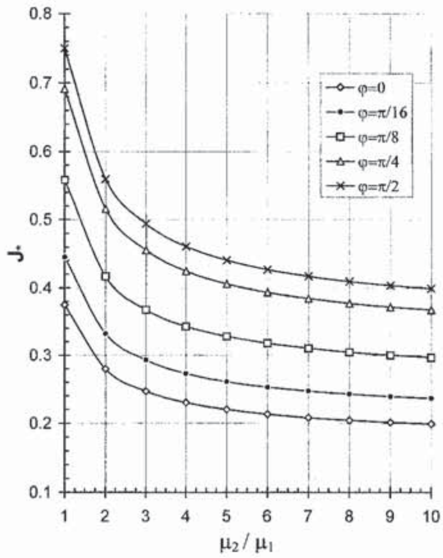

Figure 1. Nondimensional strain energy release rates versus shear moduli ratio for $a_2 : a_1 = 1: 2$; $\nu_1 = \nu_2 = 0.3$.

In Figures 1–6 the nondimensional strain energy release rate

$$J_*(\varphi) = \frac{J(\varphi)\,\mu_1}{\sigma^2 a_2} \tag{21}$$

is presented. Here $J(\varphi)$ is the strain energy release rate at a point of the crack contour defined by a polar angle φ, where

$$\varphi = \arctan\frac{x_2}{x_1} = \arctan\left(\frac{a_2}{a_1}\tan\theta\right).$$

Angle θ was introduced in (9).

Figures 1 and 2 show the dependence of J_* on the shear moduli ratio (μ_2: μ_1) at different points of the crack contour (at $\varphi = 0,\ \pi/16,\ \pi/8,\ \pi/4$ and $\pi/2$). Results are presented for the case when the Poisson's ratios are $\nu_1 = \nu_2 = 0.3$, and the semi-axes of elipse are defined as a_1: $a_2 = 2$ (Figure 1) and a_1: $a_2 = 4$ (Figure 2) respectively. Note, that for μ_2: $\mu_1 = 1$ the results are identical to the analytical solutions of the problem of an elliptical crack embedded in a homogeneous solid.

Figures 3–6 show the dependence of J_* on the angle φ for four types of nonhomogeneous solids with different material constants. Functions $J_*(\varphi)$ are presented on the interval $0 \leqslant \varphi \leqslant \pi/2$ for elliptical cracks with the following semi-axis ratios: a_1: $a_2 = 1, 2, 4, 8$ and 16.

Closed form solutions for the interface crack problems were obtained by Salganik (1963) for the case of plane strain problem, and by Mossakovskii and Rybka (1964), Willis (1972) and Kassir and Bregman (1972) for the case of penny-shaped crack. Comparison of results obtained by analytical-numerical method with the exact solutions is presented in Figures 3–6 for these two cases. In the above mentioned papers the analytical solutions are provided in different forms. To enable comparison with the present results, the values obtained by using exact expressions were transformed to the nondimensional strain energy rates J_*.

It can be observed, that for the case of penny-shaped cracks the results of the analytical-numerical method are practically identical to the analytical solutions. Note, that the exact

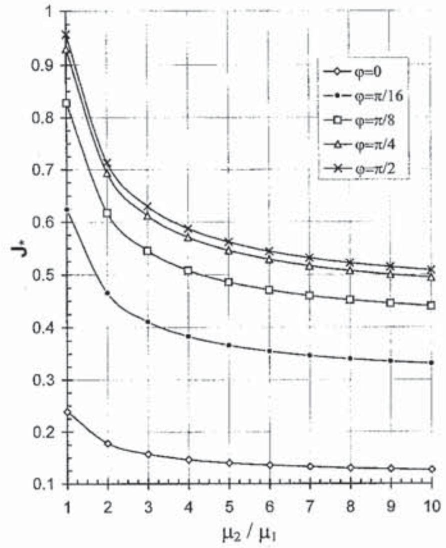

Figure 2. Nondimensional strain energy release rates versus shear moduli ratio for $a_2 : a_1 = 1{:}\ 4$; $\nu_1 = \nu_2 = 0.3$.

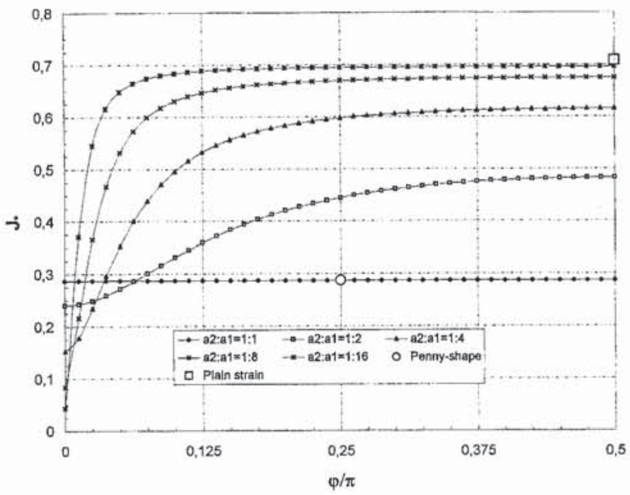

Figure 3. Nondimensional strain energy release rates for $\nu_1 = 0.2$; $\nu_2 = 0.4$; $\mu_1 : \mu_2 = 1{:}\ 4$.

solutions for function $J_*(\varphi)$ are constants for penny-shaped cracks. They are shown in Figures 3–6 only at one point (at $\varphi = \pi/4$).

It can be also seen from Figures 3–6, that the results for elongated elliptical cracks ($a_1{:}\ a_2 = 16$) at the tip of the shorter semi-axis are slightly lower of the corresponding exact plain strain solutions.

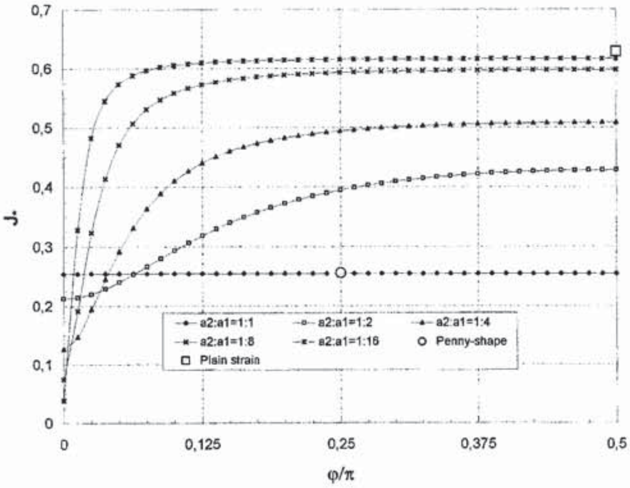

Figure 4. Nondimensional strain energy release rates for $\nu_1 = 0.2$; $\nu_2 = 0.4$; $\mu_1 : \mu_2 = 1: 10$.

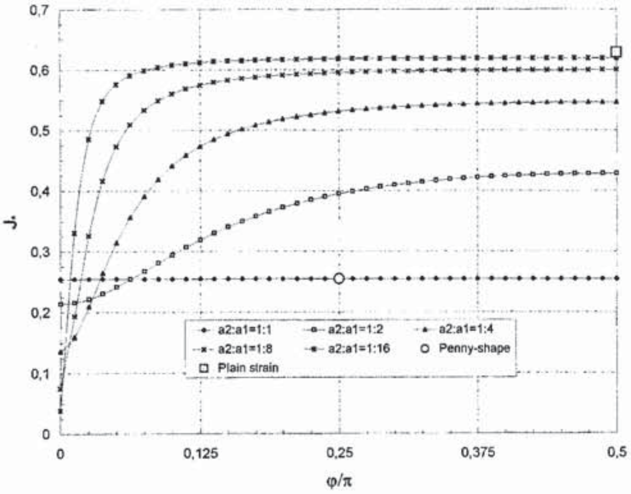

Figure 5. Nondimensional strain energy release rates for $\nu_1 = 0.4$; $\nu_2 = 0.2$; $\mu_1 : \mu_2 = 1: 4$.

Appendix

Expressions for functions $U^{pq}(x)$ and $R_{ij}^{pq}(x)$, obtained by Kaptsov and Shifrin (1995, 1996), are the following

$$
\begin{aligned}
U^{pq}(x) = {} & \frac{p!q!}{(-2)^{p+q} a_2} \sum_{r=0}^{[p/2]} \sum_{m=0}^{[q/2]} \frac{1}{r!m!\,(p-2r)!\,(q-2m)!} \\
& \times \left\{ \frac{E\,(k)}{(p+q-r-m)!} \sum_{t=0}^{p+q-r-m} \frac{\left(\frac{3}{2}\right)_t (r+m-p-q)_t}{t!} \right. \\
& \quad \times \sum_{s=0}^{t} \frac{(2s)!\,(2t-2s)!\,H\,(2s+2r-p)\,H\,(2t+2m-2s-q)}{s!\,(t-s)!\,(2s+2r-p)!\,(2t+2m-2s-q)!}
\end{aligned}
$$

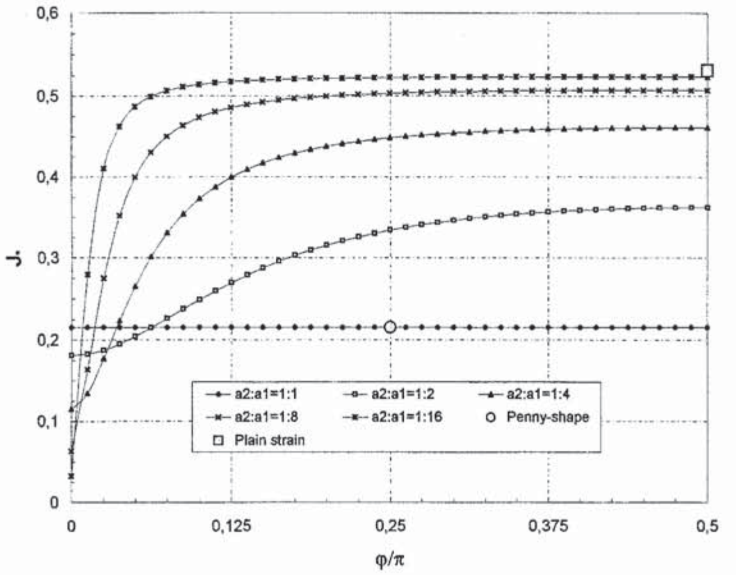

Figure 6. Nondimensional strain energy release rates for $\nu_1 = 0.4$; $\nu_2 = 0.2$; $\mu_1 : \mu_2 = 1 : 10$.

$$\times y_1^{2s+2r-p}\, y_2^{2t+2m-2s-q}$$

$$+ \sum_{l=1}^{p+q-r-m} \sum_{n=0}^{l} \frac{(-1)^{n+1}\, k^{2l}\, (2l+1)!!\, (2l)!\pi}{2^{5l}\, (l!)^2\, (p+q-r-m-l)!\, (2l-1)\, (2n)!\, (2l-2n)!}$$

$$\times {}_2F_1\left(l-\frac{1}{2}, l+\frac{1}{2}; 2l+1; k^2\right)$$

$$\times \sum_{t=0}^{p+q-r-m-l} \frac{\left(l+\frac{3}{2}\right)_t (l+r+m-p-q)_t}{(2l+1)_t}$$

$$\times \sum_{s=0}^{t} \frac{(2n+2s)!\,(2l+2t-2n-2s)!}{s!\,(t-s)!\,(2n+2s+2r-p)!\,(2l+2t+2m-2n-2s-q)!}$$

$$\times H(2n+2s+2r-p)\, H(2l+2t+2m-2n-2s-q)$$

$$\left. \times\, y_1^{2n+2s+2r-p}\, y_2^{2l+2t+2m-2n-2s-q} \right\}$$

$$R_{11}^{pq}(x) = \frac{p!q!\pi a_2}{2\,(-2)^{p+q}\, a_1^2} \sum_{r=0}^{[p/2]} \sum_{m=0}^{[q/2]} \frac{1}{r!m!\,(p-2r)!\,(q-2m)!}$$

$$\times \left\{ \frac{K(k)}{\pi\,(p+q-r-m+1)!} \sum_{t=1}^{p+q-r-m+1} \frac{\left(\frac{1}{2}\right)_t (r+m-p-q-1)_t}{t!} \right.$$

$$\times \sum_{s=1}^{t} \frac{(2s)!\,(2t-2s)!H\,(2s+2r-p-2)\,H\,(2t+2m-2s-q)}{s!\,(t-s)!\,(2s+2r-p-2)!\,(2t+2m-2s-q)!}$$

$$\times y_1^{2s+2r-p-2} y_2^{2t+2m-2s-q}$$

$$+ \sum_{l=1}^{p+q-r-m+1} \frac{(2l-1)!!\,(2l)!k^{2l}}{2^{5l}\,(l!)^2\,(p+q-r-m-l+1)!}$$

$$\times {}_2F_1\left(l+\tfrac{1}{2}, l+\tfrac{1}{2}; 2l+1; k^2\right)$$

$$\times \sum_{t=0}^{p+q-r-m-l+1} \frac{\left(l+\frac{1}{2}\right)_t (l+r+m-p-q-1)_t}{(2l+1)_t} \sum_{s=0}^{t} \frac{1}{s!\,(t-s)!}$$

$$\times \sum_{n=0}^{l} \frac{(-1)^n (2n+2s)!(2l+2t-2n-2s)!}{(2n)!(2l-2n)!(2n+2s+2r-p-2)!(2l+2t+2m-2n-2s-q)!}$$

$$\times H(2n+2s+2r-p-2)\,H(2l+2t+2m-2n-2s-q)$$

$$\left. \times\; y_1^{2n+2s+2r-p-2} y_2^{2l+2t+2m-2n-2s-q} \right\}$$

$$R_{22}^{pq}(x) \;=\; \frac{p!q!\pi}{2\,(-2)^{p+q}\,a_2} \sum_{r=0}^{[p/2]} \sum_{m=0}^{[q/2]} \frac{1}{r!m!\,(p-2r)!\,(q-2m)!}$$

$$\times \left\{ \frac{K\,(k)}{\pi\,(p+q-r-m+1)!} \right.$$

$$\times \sum_{t=1}^{p+q-r-m+1} \frac{\left(\frac{1}{2}\right)_t (r+m-p-q-1)_t}{t!}$$

$$\times \sum_{s=0}^{t} \frac{(2s)!\,(2t-2s)!\,H\,(2s+2r-p)\,H\,(2t+2m-2s-q-2)}{s!\,(t-s)!\,(2s+2r-p)!\,(2t+2m-2s-q-2)!}$$

$$\times y_1^{2s+2r-p} y_2^{2t+2m-2s-q-2}$$

$$+ \sum_{l=1}^{p+q-r-m+1} \frac{(2l-1)!!\,(2l)!k^{2l}}{2^{5l}\,(l!)^2\,(p+q-r-m-l+1)!}\;{}^{2}$$

$$\times {}_2F_1\left(l+\tfrac{1}{2}, l+\tfrac{1}{2}; 2l+1; k^2\right)$$

$$\times \sum_{t=0}^{p+q-r-m-l+1} \frac{\left(l+\frac{1}{2}\right)_t (l+r+m-p-q-1)_t}{(2l+1)_t} \sum_{s=0}^{t} \frac{1}{s!\,(t-s)!}$$

$$\times \sum_{n=0}^{l} \frac{(-1)^n (2n+2s)!(2l+2t-2n-2s)!}{(2n)!(2l-2n)!(2n+2s+2r-p)!(2l+2t+2m-2n-2s-q-2)!}$$

$$\times H(2n+2s+2r-p)\, H(2l+2t+2m-2n-2s-q-2)$$

$$\left. \times\, y_1^{2n+2s+2r-p} y_2^{2l+2t+2m-2n-2s-q-2} \right\}$$

•

$$R_{12}^{pq}(x) = \frac{p!q!\pi}{2(-2)^{p+q} a_1} \sum_{r=0}^{[p/2]} \sum_{m=0}^{[q/2]} \frac{1}{r!m!(p-2r)!(q-2m)!}$$

$$\times \left\{ \frac{K(k)}{\pi(p+q-r-m+1)!} \right.$$

$$\times \sum_{t=1}^{p+q-r-m+1} \frac{\left(\frac{1}{2}\right)_t (r+m-p-q-1)_t}{t!}$$

$$\times \sum_{s=1}^{t-1} \frac{(2s)!(2t-2s)! H(2s+2r-p-1)\, H(2t+2m-2s-q-1)}{s!(t-s)!(2s+2r-p-1)!(2t+2m-2s-q-1)!}$$

$$\times y_1^{2s+2r-p-1} y_2^{2t+2m-2s-q-1}$$

$$+ \sum_{l=1}^{p+q-r-m+1} \frac{(2l-1)!!(2l)!k^{2l}}{2^{5l}(l!)^2(p+q-r-m-l+1)!}\, {}_2F_1\left(l+\tfrac{1}{2}, l+\tfrac{1}{2}; 2l+1; k^2\right)$$

$$\times \sum_{t=0}^{p+q-r-m-l+1} \frac{\left(l+\frac{1}{2}\right)_t (l+r+m-p-q-1)_t}{(2l+1)_t} \sum_{s=0}^{t} \frac{1}{s!(t-s)!}$$

$$\times \sum_{n=0}^{l} \frac{(-1)^n (2n+2s)!(2l+2t-2n-2s)!}{(2n)!(2l-2n)!(2n+2s+2r-p-1)!(2l+2t+2m-2n-2s-q-1)!}$$

$$\times H(2n+2s+2r-p-1)\, H(2l+2t+2m-2n-2s-q-1)$$

$$\left. \times\, y_1^{2n+2s+2r-p-1} y_2^{2l+2t+2m-2n-2s-q-1} \right\}$$

In the above expressions, the square brackets denote the integer part of the value, $E(k)$ and $K(k)$ are the complete elliptical integrals, where k^2 is defined as $k^2 = (a_1^2 - a_2^2)/a_1^2$, ${}_2F_1(a, b; c; z)$ is the hypergeometric function, $(a)_n$ is the Pochhammer symbol ($(a)_0 = 1$, $(a)_n = a(a+1)\ldots(a+n-1)$), double factorial is defined as $(2l+1)!! = 1.3.5\ldots(2l+1)$, with $(-1)!! = 1$, and $H(s)$ is defined as $H(s) = \begin{cases} 1 & \text{if } s \geqslant 0 \\ 0 & \text{if } s < 0 \end{cases}$.

Function $L_1^{pq}(x)$ can be expanded into the power series as follows:

$$
\begin{aligned}
L_1^{pq}(x) &= \frac{\partial T_{pq}^{1/2}(x)}{\partial x_1} \\
&= \frac{1}{a_1}\left\{ p\, y_1^{p-1}\, y_2^q \sqrt{1-y_1^2-y_2^2} - y_1^{p+1}\, y_2^q \frac{1}{\sqrt{1-y_1^2-y_2^2}} \right\} \\
&= \frac{1}{a_1}\left\{ \sum_{n=0}^{\infty}(-1)^n \; (\tfrac{3}{2}-n)_n \sum_{k=0}^{n} \frac{p}{k!\,(n-k)!}\, y_1^{2k+p-1}\, y_2^{2n-2k+q} \right. \\
&\qquad \left. - \sum_{n=0}^{\infty}(-1)^n \; (\tfrac{1}{2}-n)_n \sum_{k=0}^{n} \frac{1}{k!\,(n-k)!}\, y_1^{2k+p+1}\, y_2^{2n-2k+q} \right\}.
\end{aligned}
$$

Analogously, the power series expression for the function $L_2^{pq}(x)$ is

$$
\begin{aligned}
L_2^{pq}(x) &= \frac{1}{a_2}\left\{ \sum_{n=0}^{\infty}(-1)^n \; (\tfrac{3}{2}-n)_n \sum_{k=0}^{n} \frac{q}{k!\,(n-k)!}\, y_1^{2k+p}\, y_2^{2n-2k+q-1} \right. \\
&\qquad \left. - \sum_{n=0}^{\infty}(-1)^n (\tfrac{1}{2}-n)_n \sum_{k=0}^{n} \frac{1}{k!\,(n-k)!}\, y_1^{2k+p}\, y_2^{2n-2k+q+1} \right\}.
\end{aligned}
$$

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