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A new finite element formulation of three-dimensional beam theory based on interpolation of curvature

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Abstract: A new finite element formulation of the 'kinematically exact finite-strain beam theory' is presented. The finite element formulation employs the generalized virtual work in which the main role is played by the pseudo-curvature vector. The solution of the governing equations is found by using a combined Galerkin-collocation algorithm.

keyword: three-dimensional beams, rotations, curvature, finite element method.

1 Introduction

Deformation of beams, plates, and shells is often characterized by large rotations. Because the spatial rotations are elements of a multiplicative group, the configuration space of deformations is a non-linear manifold. That is what makes the study of these engineering structures so interesting and challenging. Although the rotations are indeed essential for the overall deformation of the structure, they have no effect on its deformation energy. This suggests that strain measures and not rotations are natural variables for the description of the deformation energy.

The rotational strain and the rotations are related by kinematic equations in the form of differential equations. Thus, the two variables, the rotations and the rotational strain, are not independent. By the use of kinematic equations one may (at least formally) express the rotational strain measures by the rotations and thus eliminate the rotational strain as independent variable of the problem; or vice versa, the rotations can be substituted by the rotational strain. The application of the former approach where the rotations (as well as displacements) are taken to be primary variables is a typical characteristic of modern three-dimensional beam theories [Cardona and Géradin (1988); Crisfield and Jelenić (1999); Hsiao and Lin (2000, 2001); Ibrahimbegovic (1995, 1997); Iura and Atluri (1988, 1989); Jelenić and Crisfield (1999); Jelenić and Saje (1995); Li (1998); Nour-Omid and Rankin (1991); Simo (1985); Simo and Vu-Quoc (1986)]. By contrast, the present formulation employs the approach in which the rotational strain – the curvature vector – entirely replaces the rotations. For the curvature vector approximation, the additive-type of interpolation can be used without a loss of the objectivity of the strain measures, i.e., their invariance to rigid-body motions. This is in contrast to some well established finite element beam formulations [see the discussion in Crisfield and Jelenić (1999) and their solution to the problem], which are not rotational strain objective.

Another issue, also discussed by Crisfield and Jelenić (1999) and Jelenić and Crisfield (1999), is the path independence of the finite element formulation for conservative problems. Many of the established finite element formulations of three-dimensional beams are not path-independent. By way of numerical examples we show the path independence of the present formulation.

In order to apply the curvature vector as a basic variable we follow and extend the work by Planinc, Saje and Čas (2001) and propose a modified principle of virtual work for the so called 'kinematically exact finite-strain beam theory' [Simo (1985)] in which the only degree of freedom that needs to be interpolated along the element is the variation (or iterative increment) of the curvature vector. The displacement and rotational vectors are not interpolated. This 'one-field' formulation does not only result in the fact that the locking never occurs but also an enhanced accuracy for the given number of degrees of freedom is achieved. Moreover, the element enables more accurate descriptions of strain and stress distributions within the element which is of utmost importance in describing the behaviour of plastic material in the regions of localized strain.

As is well known, the stress-resultants as obtained from the equilibrium equations, and those calculated from the constitutive equations, do not equal in standard finite element formulations. The corresponding computed error in

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internal forces may be considerable, especially for materially non-linear problems. This 'inconsistency of equilibrium at cross-sections' is here solved by enforcing the consistency condition to be satisfied in a set of predefined points (here taken to coincide with the interpolation nodes) (the 'collocation'). A similar strategy was employed by Vratanar and Saje (1999) for elastic-plastic analysis of plane frames. In addition, in the present formulation, the determination of internal forces does not require the differentiation with respect to the arc-length of the beam axis, x. This is an important advantage compared to formulations where the derivatives with respect to x are needed for the evaluation of internal forces which may significantly lower the accuracy of results.

The tangent stiffness matrix and the residual force vector of a finite element are here derived with respect to the global coordinate system. The coordinate transformation from the local to global system is thus not necessary. An arbitrary initial curvature and deformation of the beam are assumed at the initial unloaded configuration.

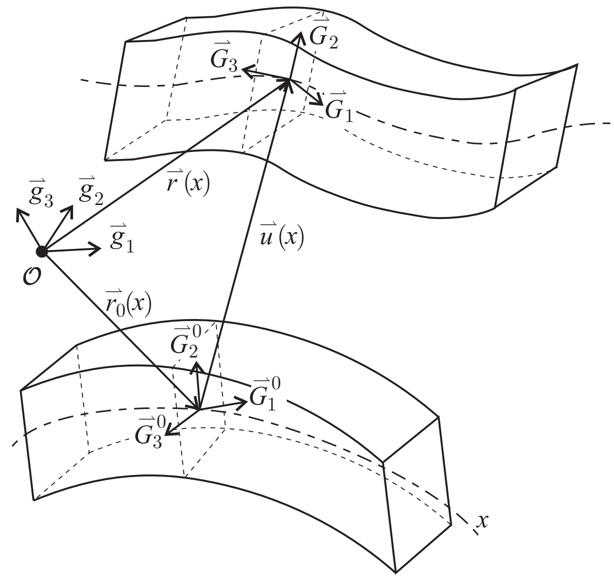

Figure 1 : Kinematics of the reference and the deformed configurations of the beam.

2 Geometry and kinematics of the beam

Geometry of the three-dimensional beam is described by the line of centroids of cross-sections and by the family of the cross-sections not necessarily normal to the line of centroids. The geometric shape of the cross-sections is assumed to be arbitrary and constant along the beam. The Bernoulli hypothesis is assumed that a cross-section suffers only rigid rotation during deformation. Two different configurations of the beam need to be distinguished:

(i) the *reference configuration* where all geometrical and mechanical variables are known;

(ii) the *deformed configuration* where the loading is prescribed while the remaining geometrical and mechanical variables are unknown.

The physical space of the motion of the beam is the Euclidean linear vector space $\mathbb{R}^3$ spanned by two orthonormal bases. The spatial (or global) basis $\left\{\vec{g}_1, \vec{g}_2, \vec{g}_3\right\}$ is an arbitrary fixed basis. Together with a reference point O the basis defines a spatial Cartesian coordinate system. The deformed configuration of the beam is described by the position vector $\vec{r}$ of the line of centroids, and by the material (or local) basis $\left\{\vec{G}_1, \vec{G}_2, \vec{G}_3\right\}$, which defines the rotated position of cross-sections. Both, the position vector $\vec{r}$, and the base vectors $\vec{G}_1$, $\vec{G}_2$, and $\vec{G}_3$, are dependent on x, the arc-length parameter of the line of centroids of cross-sections at the reference configuration. The initial, unloaded configuration will here be taken to be referential. The material basis is chosen such that the vectors $\vec{G}_2$ and $\vec{G}_3$ are directed along the principal axes of inertia of the cross-section, and that $\vec{G}_1$ is normal to the cross-section: $\vec{G}_1 = \vec{G}_2 \times \vec{G}_3$. It should be pointed out that the vector $\vec{G}_1$ is generally not parallel to the tangent vector of the line of centroids, $\vec{G}_1 \neq \vec{r}\,'_0(x)$. (Here and henceforth the prime ($'$) denotes the derivative with respect to x.) Similarly, the reference configuration is described by $\vec{r}^{\,0}$ and $\left\{\vec{G}_1^0, \vec{G}_2^0, \vec{G}_3^0\right\}$.

The reference and the current deformed material bases are related to the spatial basis by the orthogonal mapping. Let R_0 and R denote the corresponding rotation matrices; $\mathsf{R}_0(x)$ maps the basis $\left\{\vec{g}_1, \vec{g}_2, \vec{g}_3\right\}$ into the basis $\left\{\vec{G}_1^0(x), \vec{G}_2^0(x), \vec{G}_3^0(x)\right\}$, while $\mathsf{R}(x)$ maps $\left\{\vec{g}_1, \vec{g}_2, \vec{g}_3\right\}$ into $\left\{\vec{G}_1(x), \vec{G}_2(x), \vec{G}_3(x)\right\}$. A vector, $\vec{v}$, can be expressed with respect to either of the two bases

$$\vec{v} = v_{g1}\vec{g}_1 + v_{g2}\vec{g}_2 + v_{g3}\vec{g}_3 = v_{G1}\vec{G}_1 + v_{G2}\vec{G}_2 + v_{G3}\vec{G}_3.$$

For convenience, the components of the vector, $\{v_{g1}, v_{g2}, v_{g3}\}$ and $\{v_{G1}, v_{G2}, v_{G3}\}$, are also represented in the matrix form by one-column matrices

$$\boldsymbol{v}_g = \begin{bmatrix} v_{g1} \\ v_{g2} \\ v_{g3} \end{bmatrix}, \quad \boldsymbol{v}_G = \begin{bmatrix} v_{G1} \\ v_{G2} \\ v_{G3} \end{bmatrix}.$$

Both, $\boldsymbol{v}_g$ and $\boldsymbol{v}_G$, along with the corresponding basis, equivalently represent the vector $\vec{v}$. The relationship between the two one-column matrices, $\boldsymbol{v}_g$ and $\boldsymbol{v}_G$, is given by

$$\boldsymbol{v}_g = \mathsf{R}\boldsymbol{v}_G. \tag{1}$$

Here, another meaning of the rotation matrix is revealed: it rotates a vector, but it also *represents* the coordinate transformation between the components of a vector with respect to spatial and material bases.

In what follows, vectors will be replaced by one-column matrices and marked by a bold-face font. In the text they will still be termed vectors, however. E.g., the position vector $\vec{r}$ will be replaced by $\boldsymbol{r}$ and termed the position vector.

3 Stress resultants, strain measures and constitutive equations

3.1 Stress resultants and equilibrium equations

The stress-resultant force vector over the cross-section is denoted by $\boldsymbol{N}$ and the resulting moment vector by $\boldsymbol{M}$. Both, $\boldsymbol{N}$ and $\boldsymbol{M}$, are referred to the material basis. We consider a beam subjected to the external distributed force and moment vectors $\boldsymbol{n}$ and $\boldsymbol{m}$ per unit length of the reference line of centroids. $\boldsymbol{n}$ and $\boldsymbol{m}$ are taken to be given with respect to the spatial basis. The equilibrium equations of an infinitesimal element of a beam, as illustrated in Fig. 2, are given by the differential equations

$$\boldsymbol{n}(x) = -\left[\mathsf{R}(x)\boldsymbol{N}(x)\right]' \tag{2}$$

$$\boldsymbol{m}(x) = -\left[\mathsf{R}(x)\boldsymbol{M}(x)\right]' - \boldsymbol{r}'(x) \times \mathsf{R}(x)\boldsymbol{N}(x). \tag{3}$$

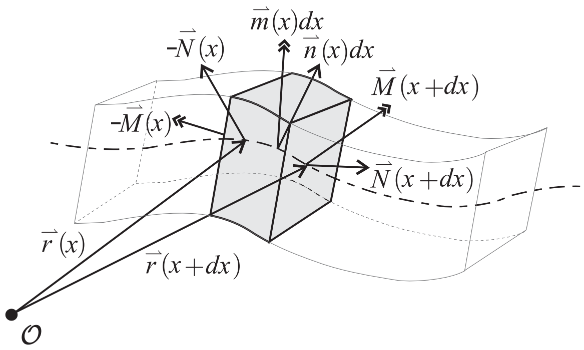

Figure 2 : The equillibrium of an infinitesimal element of a beam.

3.2 Strain measures

For an element of the beam, bounded by the cross-sections at $x = x_1$ and $x = x_2$, the principle of virtual work may be stated in the following form:

$$\int_{x_1}^{x_2} (\boldsymbol{N}\cdot\delta\boldsymbol{\gamma} + \boldsymbol{M}\cdot\delta\boldsymbol{\kappa})\,dx = \int_{x_1}^{x_2} (\boldsymbol{n}\cdot\delta\boldsymbol{r} + \boldsymbol{m}\cdot\delta\boldsymbol{\vartheta})\,dx + \left[\boldsymbol{S}\cdot\delta\boldsymbol{r} + \boldsymbol{P}\cdot\delta\boldsymbol{\vartheta}\right]_{x_1}^{x_2}. \tag{4}$$

The left-hand side of Eq. (4) determines the virtual work of internal forces, while the right-hand side defines the virtual work of external forces. In (4) $\delta\boldsymbol{r}$ and $\delta\boldsymbol{\vartheta}$ are variations of the position vector and the rotational vector of the material basis of the cross-section of the beam. In the virtual work of internal forces, two quantities are introduced that further need to be elaborated upon. These are the variations of strain vectors $\boldsymbol{\gamma}$ and $\boldsymbol{\kappa}$ with components given with respect to the material basis.

Remark 1 *Note that the variation of a one-column matrix of vector components, given with respect to the material basis, requires the variation of components only, without taking the variation of the base vectors of the material basis into account. This is in accord with the notion of 'objective rates'; see, e.g., Simo (1985).*

Inserting Eqs (2)–(3) into (4) and applying the partial integration yields the relationships between the variations of kinematic vector variables $(\boldsymbol{r}, \boldsymbol{\vartheta})$ and strain vectors $(\boldsymbol{\gamma}, \boldsymbol{\kappa})$

$$\delta\boldsymbol{\gamma} = \mathsf{R}^T\left(\delta\boldsymbol{r}' - \delta\boldsymbol{\vartheta}\times\boldsymbol{r}'\right) \tag{5}$$

$$\delta\boldsymbol{\kappa} = \mathsf{R}^T\delta\boldsymbol{\vartheta}'. \tag{6}$$

Eqs (5) and (6) relate the variations of strains, displacements, and rotations, and indicate that the variations of the quantities mentioned above are not all independent. Following the approach similar to that of Reissner (1981) and Ibrahimbegovic (1997) we can show that Eqs (5) and (6) can be integrated for strain measures $(\boldsymbol{\gamma},\boldsymbol{\kappa})$ as functions of displacements and rotations $(\boldsymbol{r},\boldsymbol{\vartheta})$, which gives

$$\boldsymbol{\gamma} = \mathsf{R}^T\boldsymbol{r}' + \boldsymbol{c} \tag{7}$$

$$\boldsymbol{\kappa} = \mathsf{R}^T\boldsymbol{\omega} + \boldsymbol{e}. \tag{8}$$

Here, vectors $\boldsymbol{c}$ and $\boldsymbol{e}$ are variational constants to be determined from the known strains and kinematics at the reference configuration by the equations

$$\boldsymbol{c} = \boldsymbol{\gamma}_0 - \mathsf{R}_0^T\boldsymbol{r}_0' \tag{9}$$

$$\boldsymbol{e} = \boldsymbol{\kappa}_0 - \mathsf{R}_0^T\boldsymbol{\omega}_0. \tag{10}$$

One-column matrix $\boldsymbol{\omega}$ introduced in (8) is the axial vector of the antisymmetric matrix $\boldsymbol{\Omega} = \mathsf{R}'\mathsf{R}^T$. Its components are given with respect to the spatial basis. In dynamics, where parameter x is replaced by time t, $\boldsymbol{\omega}$ is commonly referred to as the angular velocity. For obvious reasons, $\boldsymbol{\omega}$ is here termed the curvature. Note, however, that $\boldsymbol{\omega}$ is not the curvature of the centroid axis of the beam, so that the term 'pseudo-curvature' is here more adequate. For further descriptions of the angular velocity vector, see, e.g., Argyris (1982), Atluri and Cazzani (1995), and Crisfield (1997).

Remark 2 *The strain measures, derived in (7)–(8), are in complete agreement with those obtained by Simo (1995) and termed material strain measures (see his Eq. (4.8a)).*

For the reasons which will become clear later, an additional strain measure, $\boldsymbol{\kappa}^*$, will be introduced as follows. Vector $\delta\boldsymbol{\kappa}$ has been introduced as the energy complement to the stress-resultant $\boldsymbol{M}$. Both, $\delta\boldsymbol{\kappa}$ and $\boldsymbol{M}$, are one-column matrices of components with respect to the material basis. The spatial form of $\boldsymbol{M}$ can easily be found using Eq. (1):

$$\boldsymbol{M}_g = \mathsf{R}\boldsymbol{M}. \tag{11}$$

If we introduce the vector $\delta\boldsymbol{\kappa}^*$ by the equation

$$\delta\boldsymbol{\kappa}^* = \mathsf{R}\delta\boldsymbol{\kappa} \tag{12}$$

and put it into the scalar product $\boldsymbol{M}\cdot\delta\boldsymbol{\kappa}$, we obtain

$$\boldsymbol{M}\cdot\delta\boldsymbol{\kappa} = \boldsymbol{M}\cdot\mathsf{R}^T\delta\boldsymbol{\kappa}^* = \mathsf{R}\boldsymbol{M}\cdot\delta\boldsymbol{\kappa}^* = \boldsymbol{M}_g\cdot\delta\boldsymbol{\kappa}^*. \tag{13}$$

As observed from this result, the vector $\delta\boldsymbol{\kappa}^*$ introduced above is the energy complement to the moment $\boldsymbol{M}_g$ given with respect to the spatial basis. It is clear from the definition of $\delta\boldsymbol{\kappa}^*$ that it depends on the rigid rotation. In contrast – as it can easily be shown – the strains $\boldsymbol{\gamma}$ and $\boldsymbol{\kappa}$ do not. Despite of this shortcoming, $\delta\boldsymbol{\kappa}^*$ and $\boldsymbol{\kappa}^*$ are found to be useful in developing the beam governing equations as well as in the update procedure.

Upon inserting Eq. (12) into Eq. (6) we obtain the relationship between $\delta\boldsymbol{\kappa}^*$ and $\delta\boldsymbol{\vartheta}'$

$$\delta\boldsymbol{\kappa}^* = \delta\boldsymbol{\vartheta}'. \tag{14}$$

The integration of (14) in the sense of variations is now easy

$$\boldsymbol{\kappa}^* = \boldsymbol{\vartheta}' + \boldsymbol{d}. \tag{15}$$

Here $\boldsymbol{d}$ marks a variational constant ($\delta\boldsymbol{d} = \mathbf{0}$) to be obtained from the data in the reference configuration

$$\boldsymbol{d} = \boldsymbol{\kappa}_0^* - \boldsymbol{\vartheta}_0'. \tag{16}$$

The relationship between $\boldsymbol{\kappa}^*$ and $\boldsymbol{\kappa}$ is also needed for further use. It is obtained by employing the known relationship between $\boldsymbol{\omega}$ and $\boldsymbol{\vartheta}'$. The development of the relationship is rather lengthy and can be found, e.g., in Atluri and Cazzani (1995). The result can be written in the following form

$$\boldsymbol{\omega} = \mathsf{T}(\boldsymbol{\vartheta})\boldsymbol{\vartheta}'. \tag{17}$$

Employing (17) in (8) yields

$$\boldsymbol{\kappa} = \mathsf{R}^T\mathsf{T}\boldsymbol{\kappa}^* - \mathsf{R}^T\mathsf{T}\boldsymbol{d} + \boldsymbol{e} \tag{18}$$

$$= \mathsf{R}^T\mathsf{T}\boldsymbol{\kappa}^* + \boldsymbol{f}. \tag{19}$$

3.3 Constitutive equations

The virtual work principle assumes that $\boldsymbol{N}$ and $\boldsymbol{M}$ depend on strains $\boldsymbol{\gamma}$ and $\boldsymbol{\kappa}$. Thus, the material form of constitutive equations is needed. The constitutive law between the stress resultants and strains is taken to be given by the equations

$$\boldsymbol{N} = C_N(\boldsymbol{\gamma}-\boldsymbol{\gamma}_0, \boldsymbol{\kappa}-\boldsymbol{\kappa}_0) \tag{20}$$

$$\boldsymbol{M} = C_M(\boldsymbol{\gamma}-\boldsymbol{\gamma}_0, \boldsymbol{\kappa}-\boldsymbol{\kappa}_0). \tag{21}$$

The operators C_N and C_M must be invariant under superimposed rigid-body motions; otherwise are arbitrary operators describing material of the beam. We assume that at least the first derivatives with respect to $\boldsymbol{\gamma}$, $\boldsymbol{\kappa}$, and x of both, C_N and C_M, exist. It is clear from Eqs (20)–(21) that $\boldsymbol{N} \neq \mathbf{0}$ and $\boldsymbol{M} \neq \mathbf{0}$ could be assumed generally at the reference configuration.

4 Generalized virtual work principle

4.1 Virtual work principle

Rewriting the virtual work principle (4) for a beam of initial length L gives

$$\int_0^L (\boldsymbol{N}\cdot\delta\boldsymbol{\gamma}+\boldsymbol{M}\cdot\delta\boldsymbol{\kappa})\,dx = \int_0^L (\boldsymbol{n}\cdot\delta\boldsymbol{r}+\boldsymbol{m}\cdot\delta\boldsymbol{\vartheta})\,dx + \boldsymbol{S}^0\cdot\delta\boldsymbol{r}^0+\boldsymbol{P}^0\cdot\delta\boldsymbol{\vartheta}^0+\boldsymbol{S}^L\cdot\delta\boldsymbol{r}^L+\boldsymbol{P}^L\cdot\delta\boldsymbol{\vartheta}^L. \quad (22)$$

Here, $\boldsymbol{S}^0, \boldsymbol{P}^0, \boldsymbol{S}^L, \boldsymbol{P}^L$ are vectors of the external point loads at the boundaries $x=0$ and $x=L$. The indices 0 and L mark the value of a variable at the fixed values of the arc-length parameter $x=0$ or $x=L$. Hence, $\delta\boldsymbol{r}^0$ and $\delta\boldsymbol{r}^L$ are variations of the position vector $\boldsymbol{r}$ at $x=0$ and $x=L$, and $\delta\boldsymbol{\vartheta}^0$ and $\delta\boldsymbol{\vartheta}^L$ are variations of the rotational vector at $x=0$ and $x=L$. It should be noted that in (22) $\boldsymbol{\gamma}$, $\boldsymbol{\kappa}$, $\boldsymbol{r}$, and $\boldsymbol{\vartheta}$ are not mutually independent because kinematic conditions (7), (8), (15), and (17) hold. The only arbitrary, independent variational variables in (22) are $\delta\boldsymbol{\vartheta}$ and $\delta\boldsymbol{r}$.

4.2 Generalized virtual work principle

The four Eqs (7), (8), (15), and (17) are the constraining equations for six strain and deformation measures $\boldsymbol{\gamma}$, $\boldsymbol{\kappa}$, $\boldsymbol{\kappa}^*$, $\boldsymbol{\omega}$, $\boldsymbol{r}$, and $\boldsymbol{\vartheta}$ and their variations. Once $\boldsymbol{\kappa}$ and $\boldsymbol{\omega}$ are eliminated using (8) and (17), two independent Eqs (7) and (15) remain the constraining equations for $\boldsymbol{\gamma}$, $\boldsymbol{\kappa}^*$, $\boldsymbol{r}$ and $\boldsymbol{\vartheta}$, and their variations. By analogy with the method of Lagrangian multipliers in constrained problems of calculus of variations, the constraining equations

$$\mathsf{R}\boldsymbol{\gamma}-\boldsymbol{r}'-\mathsf{R}\boldsymbol{c}=\boldsymbol{0} \quad (23)$$

$$\boldsymbol{\kappa}^*-\boldsymbol{\vartheta}'-\boldsymbol{d}=\boldsymbol{0} \quad (24)$$

are scalarly multiplied by arbitrary, independent, at least once differentiable vector functions $\boldsymbol{a}(x)$ and $\boldsymbol{b}(x)$. The multipliers are taken to be given with respect to the spatial basis; their physical background is at the present stage of derivation not clear. The scalar products of the multipliers and the constrained equations (23) and (23) are integrated along the length of the beam

$$\int_0^L \boldsymbol{a}\cdot\left(\mathsf{R}\boldsymbol{\gamma}-\boldsymbol{r}'-\mathsf{R}\boldsymbol{c}\right)dx=0 \quad (25)$$

$$\int_0^L \boldsymbol{b}\cdot\left(\boldsymbol{\kappa}^*-\boldsymbol{\vartheta}'-\boldsymbol{d}\right)dx=0 \quad (26)$$

and varied with respect to $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{\gamma}$, $\boldsymbol{\kappa}^*$, $\boldsymbol{r}$, and $\boldsymbol{\vartheta}$

$$\int_0^L \delta\boldsymbol{a}\cdot\left(\mathsf{R}\boldsymbol{\gamma}-\boldsymbol{r}'-\mathsf{R}\boldsymbol{c}\right)dx + \int_0^L \boldsymbol{a}\cdot\left(\delta\mathsf{R}\boldsymbol{\gamma}+\mathsf{R}\delta\boldsymbol{\gamma}-\delta\boldsymbol{r}'-\delta\mathsf{R}\boldsymbol{c}\right)dx=0 \quad (27)$$

$$\int_0^L \delta\boldsymbol{b}\cdot\left(\boldsymbol{\kappa}^*-\boldsymbol{\vartheta}'-\boldsymbol{d}\right)dx + \int_0^L \boldsymbol{b}\cdot\left(\delta\boldsymbol{\kappa}^*-\delta\boldsymbol{\vartheta}'\right)dx=0. \quad (28)$$

The difference of the terms $\delta\mathsf{R}\boldsymbol{\gamma}$ and $\delta\mathsf{R}\boldsymbol{c}$ is transformed into a more useful form applying a well known formula for the variation of the rotation matrix ($\delta\mathsf{R}=\delta\boldsymbol{\Theta}\mathsf{R}$):

$$\delta\mathsf{R}\left(\boldsymbol{\gamma}-\boldsymbol{c}\right)=\delta\boldsymbol{\Theta}\mathsf{R}\left(\boldsymbol{\gamma}-\boldsymbol{c}\right)=\delta\boldsymbol{\vartheta}\times\mathsf{R}\left(\boldsymbol{\gamma}-\boldsymbol{c}\right). \quad (29)$$

The terms $\boldsymbol{a}\cdot\delta\boldsymbol{r}'$ and $\boldsymbol{b}\cdot\delta\boldsymbol{\vartheta}'$ are partially integrated and the relationship $\delta\boldsymbol{\kappa}^*=\mathsf{R}\delta\boldsymbol{\kappa}$ is employed. Then we obtain

$$\int_0^L \delta\boldsymbol{a}\cdot\left(\mathsf{R}\boldsymbol{\gamma}-\boldsymbol{r}'-\mathsf{R}\boldsymbol{c}\right)dx+\int_0^L \boldsymbol{a}\cdot\mathsf{R}\delta\boldsymbol{\gamma}\,dx + \int_0^L \boldsymbol{a}\cdot\left(\delta\boldsymbol{\vartheta}\times\mathsf{R}\left(\boldsymbol{\gamma}-\boldsymbol{c}\right)\right)dx - \left[\boldsymbol{a}\cdot\delta\boldsymbol{r}\right]_0^L+\int_0^L \boldsymbol{a}'\cdot\delta\boldsymbol{r}\,dx=0 \quad (30)$$

$$\int_0^L \delta\boldsymbol{b}\cdot\left(\boldsymbol{\kappa}^*-\boldsymbol{\vartheta}'-\boldsymbol{d}\right)dx+\int_0^L \boldsymbol{b}\cdot\mathsf{R}\delta\boldsymbol{\kappa}\,dx - \left[\boldsymbol{b}\cdot\delta\boldsymbol{\vartheta}\right]_0^L+\int_0^L \boldsymbol{b}'\cdot\delta\boldsymbol{\vartheta}\,dx=0. \quad (31)$$

By adding Eqs (30) and (31) to (22) and rearranging the terms, the following equation is derived

$$\begin{aligned}&\int_0^L \left[\delta\boldsymbol{\gamma}\cdot\left(\boldsymbol{N}-\mathsf{R}^T\boldsymbol{a}\right)+\delta\boldsymbol{\kappa}\cdot\left(\boldsymbol{M}-\mathsf{R}^T\boldsymbol{b}\right)\right]dx\\ &+\int_0^L \left[\delta\boldsymbol{\vartheta}\cdot\left(-\boldsymbol{m}-\boldsymbol{b}'+\boldsymbol{a}\times\mathsf{R}\left(\boldsymbol{\gamma}-\boldsymbol{c}\right)\right)-\delta\boldsymbol{r}\cdot\left(\boldsymbol{n}+\boldsymbol{a}'\right)\right]dx\\ &-\int_0^L \left[\delta\boldsymbol{a}\cdot\left(\mathsf{R}\boldsymbol{\gamma}-\boldsymbol{r}'-\mathsf{R}\boldsymbol{c}\right)+\delta\boldsymbol{b}\cdot\left(\boldsymbol{\kappa}^*-\boldsymbol{\vartheta}'-\boldsymbol{d}\right)\right]dx\\ &+\delta\boldsymbol{r}^0\cdot\left(\boldsymbol{S}^0+\boldsymbol{a}^0\right)+\delta\boldsymbol{\vartheta}^0\cdot\left(\boldsymbol{P}^0+\boldsymbol{b}^0\right)\\ &+\delta\boldsymbol{r}^L\cdot\left(\boldsymbol{S}^L-\boldsymbol{a}^L\right)+\delta\boldsymbol{\vartheta}^L\cdot\left(\boldsymbol{P}^L-\boldsymbol{b}^L\right)=0.\end{aligned} \quad (32)$$

In Eq. (32) the variations $\delta\boldsymbol{\gamma}$, $\delta\boldsymbol{\kappa}$, $\delta\boldsymbol{\vartheta}$, $\delta\boldsymbol{r}$, $\delta\boldsymbol{a}$, and $\delta\boldsymbol{b}$ are arbitrary and independent functions, because the constraining equations have been added. The variations $\delta\boldsymbol{r}^0$, $\delta\boldsymbol{\vartheta}^0$, $\delta\boldsymbol{r}^L$, and $\delta\boldsymbol{\vartheta}^L$ are also arbitrary and independent. As the consequence of the fundamental theorem of calculus of variations [Troutman (1983)] it follows that all the coefficients at the variations vanish and the following Euler-Lagrange equations of the three-dimensional beam are obtained

$$\boldsymbol{N}-\mathsf{R}^T\boldsymbol{a}=\boldsymbol{0} \tag{33}$$
$$\boldsymbol{M}-\mathsf{R}^T\boldsymbol{b}=\boldsymbol{0} \tag{34}$$
$$\boldsymbol{n}+\boldsymbol{a}'=\boldsymbol{0} \tag{35}$$
$$\boldsymbol{m}+\boldsymbol{b}'-\boldsymbol{a}\times\mathsf{R}\left(\boldsymbol{\gamma}-\boldsymbol{c}\right)=\boldsymbol{0} \tag{36}$$
$$\mathsf{R}\boldsymbol{\gamma}-\boldsymbol{r}'-\mathsf{R}\boldsymbol{c}=\boldsymbol{0} \tag{37}$$
$$\boldsymbol{\kappa}^*-\boldsymbol{\vartheta}'-\boldsymbol{d}=\boldsymbol{0} \tag{38}$$

together with the boundary conditions

$$\boldsymbol{S}^0+\boldsymbol{a}^0=\boldsymbol{0} \tag{39}$$
$$\boldsymbol{P}^0+\boldsymbol{b}^0=\boldsymbol{0} \tag{40}$$
$$\boldsymbol{S}^L-\boldsymbol{a}^L=\boldsymbol{0} \tag{41}$$
$$\boldsymbol{P}^L-\boldsymbol{b}^L=\boldsymbol{0}. \tag{42}$$

$\boldsymbol{N}$ and $\boldsymbol{M}$ depend on $\boldsymbol{\kappa}$ and $\boldsymbol{\gamma}$ through the constitutive equations (20) and (21). In Eqs (20)–(21), $\boldsymbol{\kappa}$ is substituted by $\boldsymbol{\kappa}^*$ and $\boldsymbol{\vartheta}$ using Eq. (18). Upon considering these additional relations, Eqs (33)–(38) constitute a system of six matrix equations for six unknown vector functions $\boldsymbol{\gamma}(x)$, $\boldsymbol{\kappa}^*(x)$, $\boldsymbol{r}(x)$, $\boldsymbol{\vartheta}(x)$, $\boldsymbol{a}(x)$, and $\boldsymbol{b}(x)$ for a given set of loads, described by $\boldsymbol{n}$, $\boldsymbol{m}$, $\boldsymbol{S}^0$, $\boldsymbol{P}^0$, $\boldsymbol{S}^L$, and $\boldsymbol{P}^L$.

Eqs (35)–(38) constitute four ordinary differential vector equations of the first order. Their solutions can be formally expressed by the following integral equations

$$\boldsymbol{a}\left(x\right)=\boldsymbol{a}^0-\int_0^x\boldsymbol{n}\left(\xi\right)d\xi \tag{43}$$
$$\begin{aligned}\boldsymbol{b}\left(x\right)=\boldsymbol{b}^0&-\int_0^x\boldsymbol{m}\left(\xi\right)d\xi\\&+\int_0^x\left[\boldsymbol{a}\left(\xi\right)\times\mathsf{R}\left(\boldsymbol{\gamma}(\xi)-\boldsymbol{c}\left(\xi\right)\right)\right]d\xi\end{aligned} \tag{44}$$
$$\boldsymbol{r}\left(x\right)=\boldsymbol{r}^0+\int_0^x\mathsf{R}\left(\boldsymbol{\gamma}\left(\xi\right)-\boldsymbol{c}\left(\xi\right)\right)d\xi \tag{45}$$
$$\boldsymbol{\vartheta}\left(x\right)=\boldsymbol{\vartheta}^0+\int_0^x\left(\boldsymbol{\kappa}^*\left(\xi\right)-\boldsymbol{d}\left(\xi\right)\right)d\xi. \tag{46}$$

Eqs (45) and (46) represent the relationship between the deformation and kinematic variables. Eqs (43) and (44) are the force and moment equilibrium conditions. The physical meaning of the Lagrangian multipliers $\boldsymbol{a}$ and $\boldsymbol{b}$ is now obvious from (43) and (44): $\boldsymbol{a}(x)$ is the cross-sectional force resultant at point x; $\boldsymbol{b}(x)$ is the cross-sectional moment resultant at point x, both given with respect to the spatial basis. We have already introduced the cross-sectional force and moment resultants as computed from the strains by the constitutive equations (20)–(21). These resultants are termed the *constitutive* force and moment, $\boldsymbol{N}_C$ and $\boldsymbol{M}_C$, respectively. By contrast, the cross-sectional force and moment resultants, $\boldsymbol{a}$ and $\boldsymbol{b}$, satisfy the equilibrium equations and will hence be referred to as the *equilibrium* force and moment. Thus, Eqs (33) and (34) require that the equilibrium force and moment vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are equal to the constitutive force and moment vectors $\boldsymbol{N}$ and $\boldsymbol{M}$, respectively. These conditions yield the so called ‘consistent equilibrium at the cross-section’. For an application of these important consistency conditions in the elastic-plastic finite element analysis of plane frames, see the paper by Vratanar and Saje (1999).

Remark 3 *The discrepancy of equilibrium and constitutive forces and moments (or more generally, stresses) is a common characteristic of standard displacement-based finite element formulations. It may be a substantial source of error of a method especially in materially non-linear problems. The present formulation enforces the consistency condition to be satisfied in a set of predefined points of the centroid axis. The points are taken to coincide with the interpolation nodes.*

Let us take that the set of Eqs (43)–(46) is exactly satisfied when $\boldsymbol{n}$, $\boldsymbol{m}$, $\boldsymbol{\gamma}$, $\boldsymbol{\vartheta}$, and $\boldsymbol{\kappa}$ are known at any point of the line of centroids. We further assume that the strain vector $\boldsymbol{\gamma}$ can uniquely be determined from (20) provided $\boldsymbol{\kappa}$, $\boldsymbol{\kappa}_0$, and $\boldsymbol{\gamma}_0$ are known. As a result, Eq. (34) remains the only equation of the system (33)–(38) that still needs to be solved. The virtual work principle (32) then reduces to

$$\begin{aligned}&\int_0^L\delta\boldsymbol{\kappa}\cdot\left(\boldsymbol{M}-\mathsf{R}^T\boldsymbol{b}\right)dx\\&+\delta\boldsymbol{r}^0\cdot\left(\boldsymbol{S}^0+\boldsymbol{a}^0\right)+\delta\boldsymbol{\vartheta}^0\cdot\left(\boldsymbol{P}^0+\boldsymbol{b}^0\right)\\&+\delta\boldsymbol{r}^L\cdot\left(\boldsymbol{S}^L-\boldsymbol{a}^L\right)+\delta\boldsymbol{\vartheta}^L\cdot\left(\boldsymbol{P}^L-\boldsymbol{b}^L\right)=0.\end{aligned} \tag{47}$$

In the reduced principle of virtual work (47), the functional is dependent on the function $\boldsymbol{\kappa}(x)$ and on a set of boundary values. $\boldsymbol{a}^L$ and $\boldsymbol{b}^L$ are computed using (43)–(44):

$$\boldsymbol{a}(L) = \boldsymbol{a}^0 - \int_0^L \boldsymbol{n}(\xi)\, d\xi \tag{48}$$

$$\boldsymbol{b}(L) = \boldsymbol{b}^0 - \int_0^L \boldsymbol{m}(\xi)\, d\xi + \int_0^L [\boldsymbol{a}(\xi) \times \mathsf{R}(\boldsymbol{\gamma}(\xi) - \boldsymbol{c}(\xi))]\, d\xi. \tag{49}$$

In order to embed the element of the beam into the physical space, the boundary conditions at $x = L$ for $\boldsymbol{r}$ and $\boldsymbol{\vartheta}$ have to be specified using Eqs (45) and (46):

$$\boldsymbol{r}(L) = \boldsymbol{r}^0 + \int_0^L \mathsf{R}(\boldsymbol{\gamma}(\xi) - \boldsymbol{c}(\xi))\, d\xi \tag{50}$$

$$\boldsymbol{\vartheta}(L) = \boldsymbol{\vartheta}^0 + \int_0^L (\boldsymbol{\kappa}^*(\xi) - \boldsymbol{d}(\xi))\, d\xi. \tag{51}$$

Employing (48)–(51) and considering that the variations in (47) are arbitrary, yields the final form of the governing equations of the three-dimensional beam:

$$\boldsymbol{M}_G(x) - \mathsf{R}^T \boldsymbol{b}_g(x) = \boldsymbol{0} \tag{52}$$

$$\boldsymbol{r}_g^L - \boldsymbol{r}_g^0 - \int_0^L \mathsf{R}(\boldsymbol{\gamma}_G - \boldsymbol{c}_G)\, dx = \boldsymbol{0} \tag{53}$$

$$\boldsymbol{\vartheta}_g^L - \boldsymbol{\vartheta}_g^0 - \int_0^L \left(\boldsymbol{\kappa}_g^* - \boldsymbol{d}_g\right) dx = \boldsymbol{0} \tag{54}$$

$$\boldsymbol{S}_g^0 + \boldsymbol{a}_g^0 = \boldsymbol{0} \tag{55}$$

$$\boldsymbol{P}_g^0 + \boldsymbol{b}_g^0 = \boldsymbol{0} \tag{56}$$

$$\boldsymbol{S}_g^L - \boldsymbol{a}_g^0 + \int_0^L \boldsymbol{n}_g\, dx = \boldsymbol{0} \tag{57}$$

$$\boldsymbol{P}_g^L - \boldsymbol{b}_g^0 - \int_0^L \left[\boldsymbol{a}_g \times \mathsf{R}(\boldsymbol{\gamma}_G - \boldsymbol{c}_G) - \boldsymbol{m}_g\right] dx = \boldsymbol{0}. \tag{58}$$

Eqs (52)–(58) and auxiliary relations (59)–(66)

$$\boldsymbol{a}_g(x) = \boldsymbol{a}_g^0 - \int_0^x \boldsymbol{n}_g(\xi)\, d\xi \tag{59}$$

$$\boldsymbol{b}_g(x) = \boldsymbol{b}_g^0 - \int_0^x \boldsymbol{m}_g(\xi)\, d\xi + \int_0^x \left[\boldsymbol{a}_g(\xi) \times \mathsf{R}(\xi)\, [\boldsymbol{\gamma}_G(\xi) - \boldsymbol{c}_G(\xi)]\right] d\xi \tag{60}$$

$$\boldsymbol{r}_g(x) = \boldsymbol{r}_g^0 + \int_0^x \mathsf{R}(\xi)\, [\boldsymbol{\gamma}_G(\xi) - \boldsymbol{c}_G(\xi)]\, d\xi \tag{61}$$

$$\boldsymbol{\vartheta}_g(x) = \boldsymbol{\vartheta}_g^0 + \int_0^x \left[\boldsymbol{\kappa}_g^*(\xi) - \boldsymbol{d}_g(\xi)\right] d\xi \tag{62}$$

$$\boldsymbol{\kappa}_G = \mathsf{R}^T \mathsf{T} \boldsymbol{\kappa}_g^* + \boldsymbol{f}_G \tag{63}$$

$$\boldsymbol{N}_G = \mathsf{R}^T \boldsymbol{a}_g \tag{64}$$

$$\boldsymbol{N}_G = C_N(\boldsymbol{\gamma}_G - \boldsymbol{\gamma}_{G,0}, \boldsymbol{\kappa}_G - \boldsymbol{\kappa}_{G,0}) \tag{65}$$

$$\boldsymbol{M}_G = C_M(\boldsymbol{\gamma}_G - \boldsymbol{\gamma}_{G,0}, \boldsymbol{\kappa}_G - \boldsymbol{\kappa}_{G,0}) \tag{66}$$

constitute the complete set of equations of the three-dimensional beam. The indices, indicating the basis used, are added for the clarity of notation in these equations.

5 Finite element formulation

5.1 Component form of governing equations

To develop the algorithm for the numerical solution of the system of equations (52)–(58), the component forms of equations are needed. They read

$$f_i(x) = M_i(x) - \mathrm{R}_{ji}(x)\, b_j(x) = 0 \tag{67}$$

$$h_i = r_i^L - r_i^0 - \int_0^L \mathrm{R}_{ij}(\gamma_j - c_j)\, dx = 0 \tag{68}$$

$$h_{3+i} = \vartheta_i^L - \vartheta_i^0 - \int_0^L (\kappa_i^* - d_i)\, dx = 0 \tag{69}$$

$$h_{6+i} = S_i^0 - a_i^0 = 0 \tag{70}$$

$$h_{9+i} = P_i^0 - b_i^0 = 0 \tag{71}$$

$$h_{12+i} = S_i^L - a_i^0 + \int_0^L n_i\, dx = 0 \tag{72}$$

$$h_{16+i} = P_i^L - b_i^0 + \int_0^L m_i\, dx - \int_0^L e_{imn} a_m \mathrm{R}_{nj}(\gamma_j - c_j)\, dx = 0. \tag{73}$$

Here, indices i, j, k, l, m, n, r take the values $1,2,3$. The summation convention is used that the repeated index is the summation index. The components c_i and d_i of vectors $\boldsymbol{c}$ and $\boldsymbol{d}$ are defined at the reference configuration of the beam. The components a_i, γ_i, M_i, b_i are determined from Eqs (59)–(66) given in the component form

$$a_i(x) = a_i^0 - \int_0^x n_i\, d\xi \tag{74}$$

$$b_i(x) = b_i^0 + \int_0^x [e_{irn} a_r \mathrm{R}_{nj}(\gamma_j - c_j) - m_i]\, d\xi \tag{75}$$

$$r_i(x) = r_i^0 + \int_0^x \mathrm{R}_{ij}(\gamma_j - c_j)\, d\xi \tag{76}$$

$$\vartheta_i(x) = \vartheta_i^0 + \int_0^x (\kappa_i^* - d_i)\, d\xi \tag{77}$$

$$\kappa_i = \mathrm{R}_{ji} \mathrm{T}_{jk} \kappa_k^* + f_i \tag{78}$$

$$N_i = \mathrm{R}_{ji} a_j \tag{79}$$

$$N_i = C_i^N(\gamma_1 - \gamma_{1,0}, \ldots, \kappa_3 - \kappa_{3,0}) \tag{80}$$

$$M_i = C_i^M(\gamma_1 - \gamma_{1,0}, \ldots, \kappa_3 - \kappa_{3,0}). \tag{81}$$

Symbol e_{ijk} is the permutational symbol [Sokolnikoff (1951)]. The components of the rotation matrix, R_{ij}, are

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determined by the Rodrigues formula

$$\mathrm{R}_{ij} = \mathrm{I}_{ij} + \frac{\sin\vartheta}{\vartheta}\Theta_{ij} + \frac{1-\cos\vartheta}{\vartheta^2}\Theta_{ik}\Theta_{kj}, \tag{82}$$

where I_{ij} are the components of the unit matrix, $\vartheta = \sqrt{\vartheta_1^2+\vartheta_2^2+\vartheta_3^2}$, and Θ_{ij} is defined by

$$\begin{aligned} \Theta_{ij} &= 0, \quad \text{if} \quad i = j \\ \Theta_{12} &= -\Theta_{21} = -\vartheta_3 \\ \Theta_{13} &= -\Theta_{31} = \vartheta_2 \\ \Theta_{23} &= -\Theta_{32} = -\vartheta_1. \end{aligned}$$

Recall that $\vec{\vartheta} = \vartheta_1\vec{g}_1 + \vartheta_2\vec{g}_2 + \vartheta_3\vec{g}_3$. The components of the transformation matrix $\mathbf{T}$ are given by [Atluri and Cazzani (1995)]

$$\mathrm{T}_{ij} = \mathrm{I}_{ij} + \frac{1-\cos\vartheta}{\vartheta^2}\Theta_{ij} + \frac{\vartheta-\sin\vartheta}{\vartheta^3}\Theta_{ik}\Theta_{kj}. \tag{83}$$

Remark 4 *Eqs (82) and (83) have a singularity point at $\vartheta = 0$. The singularity can be eliminated in the following way. When ϑ equals to zero it follows that the rotation and the transformation matrix are unit matrices. In numerical calculations a strict use of Eqs (82) and (83) would lead to indefinite expressions, if ϑ were less than the machine precision. However, when ϑ is replaced by the value of the machine precision, Eqs (82) and (83) appear to be evaluated exactly with respect to the finite precision arithmetic of the computer.*

5.2 Discretization of governing equations

The arguments of the integrals in Eqs (68)–(69) and (72)–(73) are too complicated for the analytical solution to be possible; therefore, the numerical integration is introduced where the integrations are substituted by finite sums over the *global* integration nodes x_p

$$\int_0^L f(x)\,dx \to \sum_{p=1}^{N} w_p f(x_p), \tag{84}$$

which introduces an error of the discretization method. The values of the weights, w_p, and the abscisae, x_p, of nodes where function f is to be evaluated, are dependent on their number N and on the quadrature rule used. For the sake of simplicity, we will omit the summation operator and write

$$\sum_{p=1}^{N} w_p f(x_p) \to w_p f^p \tag{85}$$

where f^p denotes the value of the function at the global integration node x_p. The integrals of the external distributed force and moment vectors are, without the loss of generality, assumed to be evaluated analytically. The notation

$$n_i^L = \int_0^L n_i(x)\,dx \tag{86}$$

$$m_i^L = \int_0^L m_i(x)\,dx \tag{87}$$

will be used in the sequel.

Making use of the numerical integration, we get a discrete form of (68)–(73)

$$\tilde{h}_i = r_i^L - r_i^0 - w_p \mathrm{R}_{ij}^p\left(\gamma_j^p - c_j^p\right) = 0 \tag{88}$$

$$\tilde{h}_{3+i} = \vartheta_i^L - \vartheta_i^0 - w_p\left(\kappa_i^{*p} - d_i^p\right) = 0 \tag{89}$$

$$\tilde{h}_{6+i} = S_i^0 + a_i^0 = 0 \tag{90}$$

$$\tilde{h}_{9+i} = P_i^0 + b_i^0 = 0 \tag{91}$$

$$\tilde{h}_{12+i} = S_i^L - a_i^0 + n_i^L = 0 \tag{92}$$

$$\begin{aligned} \tilde{h}_{16+i} &= P_i^L - b_i^0 + m_i^L \\ &\quad - w_p\left[e_{irn}a_r^p\left(\mathrm{R}_{nj}^p\left(\gamma_j^p - c_j^p\right)\right)\right] = 0. \end{aligned} \tag{93}$$

The selection of the positions of the integration nodes also concerns the way Eq. (67) is discretized. That is, we require that Eq. (67) is satisfied at integration nodes only:

$$\begin{aligned} & f_i(x_p) = \tilde{h}_{17+p+i} = M_i(x_p) - \mathrm{R}_{ji}(x_p)\, b_j(x_p) = 0, \\ & \quad i = 1,2,3; \quad p = 1,2,\ldots,N. \end{aligned} \tag{94}$$

The resulting discretized system constitutes a system of $18+3N$ non-linear algebraic equations of the beam element for the unknowns r_1^0, r_2^0, r_3^0, ϑ_1^0, ϑ_2^0, ϑ_3^0, a_1^0, a_2^0, a_3^0, b_1^0, b_2^0, b_3^0, r_1^L, r_2^L, r_3^L, ϑ_1^L, ϑ_2^L, ϑ_3^L, κ_1^{*p}, κ_2^{*p}, κ_3^{*p} $(p = 1,2,\ldots,N)$.

In order to determine values of the dependent variables $a_i(x)$, $b_i(x)$, $r_i(x)$, and $\vartheta_i(x)$ at nodes x_p from the nodal curvatures κ_1^{*p}, κ_2^{*p}, κ_3^{*p}, a set of additional *local* (or internal) integrals needs to be evaluated numerically. A low order local integration which would employ only global integration nodes could be used. However, we do not wish to restrict ourselves by the order of the numerical integration. For that purpose, some interpolation of the nodal curvatures must be introduced. The space of the curvature vectors is clearly non-linear. The consequences of non-linearity are discussed in a greater detail in the section on the update procedure in Newton's iteration. At this point it suffices to say that the non-linearity of the curvature vector forced us to interpolate the variations of κ_i^*, i.e.

$$\delta\kappa_i^*(x) = I_p(x)\,\delta\kappa_i^{*p}. \tag{95}$$

The space of the variations of κ_i^* is found to be linear, so the additive-type of interpolation is correct. $I_p(x)$ are the interpolation functions (not necessarily polynomials and not necessarily continuous functions) through the integration points x_p, such that

$$I_p(x_q) = \delta_{pq} = \begin{cases} 1, & p = q \\ 0, & p \neq q \end{cases}. \tag{96}$$

The interpolation of $\delta\kappa_i^*$ and not of κ_i^* is crucial and has a direct influence on Eq. (89), which should be recast in its variational form

$$\tilde{h}_{3+i} = \delta\vartheta_i^L - \delta\vartheta_i^0 - w_p\delta\kappa_i^{*p} = 0. \tag{97}$$

Remark 5 *Replacing Eq. (89) with its weak form (97) still achieves its goal, as any increment $\delta\kappa_i^*$ preserves kinematically exact boundary incremental rotations.*

The local integrals over closed intervals $[0,x_p]$ are replaced by quadrature rules of order N_p. The interpolation of discrete values $\delta\kappa_i^{*p}$ allows us to introduce the numerical integration of any order. The quadrature rules used are formulated as

$$\int_0^{x_p} h(\xi)\,d\xi \rightarrow \sum_{s_p=1}^{N^p} \tilde{w}_{s_p} h\left(y_{s_p}\right) \tag{98}$$

with $\tilde{w}_{s_p}$ being the weights and y_{s_p} abscisae of the quadrature for any fixed interval $[0,x_p]$. The range of index s_p is taken to be $1,\ldots,N_p$, where N_p denotes the number of points of the local numerical integration. Note that N_p depends on 'p'.

Remark 6 *In the computer programming of the algorithm, an effective step-by-step computation of the local integrals can be implemented. In each integration step, solely the quadrature between the two subsequent global integration nodes is applied and the result is added to the previously obtained one. Unfortunately, such algorithm can not be conveniently written in a simple expression. Hence, for simplicity, the form as in Eq. (98) will instead be used in the text.*

An example showing an element using 4-node global Gaussian integration ($N = 4$), and 3-point local Gaussian integration between the global integration nodes is illustrated in Fig. 3. Note that we need no local points between the last global node, x_N, and the right boundary point, because, for the integration over the whole element, solely the global integration nodes are needed.

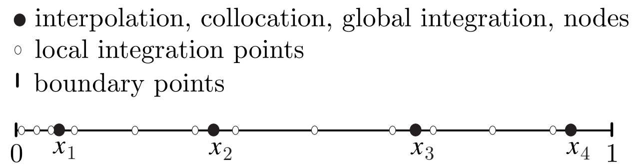

Figure 3 : The interpolation, the collocation and the global integration nodes; and the local integration points.

Remark 7 *Observe that the Galerkin-type of the finite element method would employ the interpolation of functions $\delta\kappa_i(x)$, as given in (95), in the virtual work principle (47). In such manner, one would obtain from (47) N integral equations*

$$\int_0^L (M_i - \mathrm{R}_{ji}b_j)\,I_p\,dx = 0, \quad p = 1,2,\ldots,N,$$

which would be replaced by the summations using the numerical integration

$$w_q\left(M_i(x_q) - \mathrm{R}_{ji}(x_q)\,b_j(x_q)\right)I_p(x_q) = 0, \quad p = 1,2,\ldots,N.$$

When choosing the interpolation through the integration points, it follows $I_p(x_q) = \delta_{pq}$, which yields

$$M_i(x_p) - \mathrm{R}_{ji}(x_p)\,b_i(x_p) = 0.$$

This is exactly the same result as that given in (94) and obtained without employing the interpolation functions.

5.3 Linearization of discretized equilibrium equations

In order to find the solution of the system of non-linear algebraic equations (88)–(94), Newton's method is used. The crucial step of the method is the generation of the Jacobian matrix of the system, commonly referred to as the tangent stiffness matrix. Therefore, the partial derivatives of functions $\tilde{h}_i$, ..., $\tilde{h}_{18+3N}$ with respect to all unknowns need to be obtained. The deduction of the derivatives is greatly simplified if the partial derivatives of the quantities present in the equations are prepared in advance:

$$\begin{aligned}
&\frac{\partial a_i(x)}{\partial a_i^0} = 1\\
&\frac{\partial \vartheta_i(x)}{\partial \vartheta_i^0} = 1\\
&\frac{\partial \vartheta_i(x)}{\partial \kappa_i^{*q}} = Q_q(x)\\
&\frac{\partial \mathrm{R}_{ij}(x)}{\partial \vartheta_m^0} = -e_{ikm}\mathrm{R}_{kj}(x)\\
&\frac{\partial \mathrm{R}_{ij}(x)}{\partial \kappa_m^{*q}} = -e_{ikm}Q_q(x)\,\mathrm{R}_{kj}(x)\\
&\frac{\partial \kappa_i(x)}{\partial \kappa_j^{*q}} = \mathrm{R}_{ji}(x)\,I_q(x)\\
&\frac{\partial \gamma_k(x)}{\partial \vartheta_m^0} = -\tilde{\mathrm{C}}_{ki}(x)\,e_{jnm}\mathrm{R}_{ni}(x)\,a_j(x)\\
&\frac{\partial \gamma_k(x)}{\partial a_j^0} = \tilde{\mathrm{C}}_{ki}(x)\,\mathrm{R}_{ji}(x)\\
&\frac{\partial \gamma_k(x)}{\partial \kappa_m^{*q}} = -\tilde{\mathrm{C}}_{ki}(x)\,\mathrm{C}_{i,r+3}(x)\,\mathrm{R}_{mr}(x)\,I_q(x)\\
&\qquad\qquad - \tilde{\mathrm{C}}_{ki}(x)\,e_{jnm}Q_q(x)\,\mathrm{R}_{ni}(x)\,a_j(x)\,.
\end{aligned} \tag{99}$$

The proof of the relations (99) is found in Appendix A. The components C_{ij} introduced above stem from the linearization of the constitutive equations and read

$$\begin{aligned}
\mathrm{C}_{ij} &= \frac{\partial C_i^N}{\partial \gamma_j}, \quad \mathrm{C}_{i,j+3} = \frac{\partial C_i^N}{\partial \kappa_j}\\
\mathrm{C}_{i+3,j} &= \frac{\partial C_i^M}{\partial \gamma_j}, \quad \mathrm{C}_{i+3,j+3} = \frac{\partial C_i^M}{\partial \kappa_j}.
\end{aligned} \tag{100}$$

The notation $\tilde{\mathrm{C}}_{ki}$ is used for the components of the inverse of the matrix $\mathsf{C}_{\gamma\gamma} = [\mathrm{C}_{ij}]_{i,j=1,2,3}$. Vector components $Q_q(x)$ designate the integrals of the interpolation functions ($Q_q(x) = \int_0^x I_q(\xi)\,d\xi$). The non-zero partial derivatives of the discrete governing equations are as follows:

$$\frac{\partial \tilde{h}_i}{\partial r_i^0} = -1 \quad \frac{\partial \tilde{h}_i}{\partial r_i^L} = 1$$

$$\frac{\partial \tilde{h}_i}{\partial \vartheta_k^0} = -w_p\left[\frac{\partial \mathrm{R}_{ij}^p}{\partial \vartheta_k^0}\left(\gamma_j^p - c_j^p\right) + \mathrm{R}_{ij}^p\frac{\partial \gamma_j^p}{\partial \vartheta_k^0}\right]$$

$$\frac{\partial \tilde{h}_i}{\partial a_k^0} = -w_p\mathrm{R}_{ij}^p\frac{\partial \gamma_j^p}{\partial a_k^0}$$

$$\frac{\partial \tilde{h}_i}{\partial \kappa_m^{*q}} = -w_p\left[\frac{\partial \mathrm{R}_{ij}^p}{\partial \kappa_m^{*q}}\left(\gamma_j^p - c_j^p\right) + \mathrm{R}_{ij}^p\frac{\partial \gamma_j^p}{\partial \kappa_m^{*q}}\right]$$

$$\frac{\partial \tilde{h}_{3+i}}{\partial \vartheta_i^0} = -1 \qquad \frac{\partial \tilde{h}_{3+i}}{\partial \vartheta_i^L} = 1 \qquad \frac{\partial \tilde{h}_{3+i}}{\partial \kappa_i^{*p}} = -w_p$$

$$\frac{\partial \tilde{h}_{6+i}}{\partial a_i^0} = 1 \qquad \frac{\partial \tilde{h}_{9+i}}{\partial b_i^0} = 1 \qquad \frac{\partial \tilde{h}_{12+i}}{\partial a_i^0} = -1$$

$$\frac{\partial \tilde{h}_{16+i}}{\partial \vartheta_k^0} = -w_p\left[e_{irn}a_r^p\frac{\partial \mathrm{R}_{nj}^p}{\partial \vartheta_k^0}\left(\gamma_j^p - c_j^p\right) + e_{irn}a_r^p\mathrm{R}_{nj}^p\frac{\partial \gamma_j^p}{\partial \vartheta_k^0}\right]$$

$$\frac{\partial \tilde{h}_{16+i}}{\partial a_k^0} = -w_p\left[e_{irn}\delta_{rk}\mathrm{R}_{nj}^p\left(\gamma_j^p - c_j^p\right) + e_{irn}a_r^p\mathrm{R}_{nj}^p\frac{\partial \gamma_j^p}{\partial a_k^0}\right]$$

$$\frac{\partial \tilde{h}_{16+i}}{\partial b_i^0} = -1$$

$$\frac{\partial \tilde{h}_{16+i}}{\partial \kappa_m^{*q}} = -w_p\left[e_{irn}a_r^p\frac{\partial \mathrm{R}_{nj}^p}{\partial \kappa_m^{*q}}\left(\gamma_j^p - c_j^p\right) + e_{irn}a_r^p\mathrm{R}_{nj}^p\frac{\partial \gamma_j^p}{\partial \kappa_m^{*q}}\right].$$

By substituting relations (99) into the above partial derivatives, we obtain the components of the tangent stiffness matrix of an element. A more precise observation is taken only upon the derivatives of functions $\tilde{h}_{17+p+i}$, $p = 1,\ldots,N$. Employing the numerical integration in place of the analytical for $b_j(x_p)$ yields

$$\tilde{h}_{17+p+i} = C_i^M - \mathrm{R}_{ji}^p\left[b_j^0 - \tilde{w}_{s_p}e_{jrn}a_r^{s_p}\mathrm{R}_{nl}^{s_p}\left(\gamma_l^{s_p} - c_l^{s_p}\right) - m_j^p\right].$$

With the help of Eqs (100) we obtain

$$\begin{aligned}
\frac{\partial \tilde{h}_{17+p+i}}{\partial \vartheta_k^0} &= \mathrm{C}_{3+i,j}^p\frac{\partial \gamma_j^p}{\partial \vartheta_k^0} - \frac{\partial \mathrm{R}_{ji}^p}{\partial \vartheta_k^0}b_j^p\\
&+\mathrm{R}_{ji}^p\tilde{w}_{s_p}\left[e_{jrn}a_r^{s_p}\frac{\partial \mathrm{R}_{nl}^{s_p}}{\partial \vartheta_k^0}\left(\gamma_l^{s_p} - c_l^{s_p}\right) + e_{jrn}a_r^{s_p}\mathrm{R}_{nl}^{s_p}\frac{\partial \gamma_l^{s_p}}{\partial \vartheta_k^0}\right]\\
\frac{\partial \tilde{h}_{17+p+i}}{\partial a_k^0} &= \mathrm{C}_{3+i,j}^p\frac{\partial \gamma_j^p}{\partial a_k^0}\\
&+\mathrm{R}_{ji}^p\tilde{w}_{s_p}\left[e_{jrn}\delta_{nk}\mathrm{R}_{rl}^{s_p}\left(\gamma_l^{s_p} - c_l^{s_p}\right) + e_{jrn}a_r^{s_p}\mathrm{R}_{nl}^{s_p}\frac{\partial \gamma_l^{s_p}}{\partial a_k^0}\right]
\end{aligned} \tag{101}$$

$$\frac{\partial \tilde{h}_{17+p+i}}{\partial b_j^0} = -\mathrm{R}_{ji}^p$$

$$\begin{aligned}
\frac{\partial \tilde{h}_{17+p+i}}{\partial \kappa_m^{*q}} &= \mathrm{C}_{3+i,j}^p\frac{\partial \gamma_j^{*p}}{\partial \kappa_m^q} + \mathrm{C}_{3+i,3+j}^p\frac{\partial \kappa_j^p}{\partial \kappa_m^{*q}} - \frac{\partial \mathrm{R}_{ji}^p}{\partial \kappa_m^{*q}}b_j^p\\
&+\mathrm{R}_{ji}^p\tilde{w}_{s_p}\left[e_{jrn}a_r^{s_p}\frac{\partial \mathrm{R}_{nl}^{s_p}}{\partial \kappa_m^{*q}}\left(\gamma_l^{s_p} - c_l^{s_p}\right) + e_{jrn}a_r^{s_p}\mathrm{R}_{nl}^{s_p}\frac{\partial \gamma_l^{s_p}}{\partial \kappa_m^{*q}}\right].
\end{aligned}$$

On substituting (99) into (101), the result is completed. The global tangent stiffness matrix of a structure is obtained by assembling of tangent stiffness matrices of all beam elements. It, however, should be emphasized, that the tangent stiffness matrix of an element is described in the global (spatial) coordinate system from the outset, so that no coordinate transformation from the local to the global coordinate system is needed.

5.4 The update procedure

Following Newton's iteration scheme, at each iteration step $n = 0, 1, 2, \ldots$ a system of linear equations is solved

$$\mathsf{K}^{[n]}\Delta\boldsymbol{y} = -\boldsymbol{h}^{[n]}, \tag{102}$$

where $\mathsf{K}^{[n]}$ denotes the global tangent stiffness matrix, $\boldsymbol{h}^{[n]}$ is the vector of functions $\tilde{h}_i$, ..., $\tilde{h}_{18+3N}$, both in iteration n, and $\Delta\boldsymbol{y}$ is a vector of corrections, which is in classical Newton's method in linear vector spaces added to the previous solution iterate vector $\boldsymbol{y}^{[n]}$ of the non-linear problem. Unfortunately, the configuration space of the three-dimensional rotations is not linear, which does not allow us to sum the corrections of rotational vectors and other quantities connected with rotational vectors.

As a result of an iteration step, the corrections of the unknowns are obtained, Δr_i^0, $\Delta\vartheta_i^0$, Δa_i^0, Δb_i^0, Δr_i^L, $\Delta\vartheta_i^L$, $\Delta\kappa_i^{*p}$ $(p = 1, 2, \ldots, N)$. New values of the position vector and stress resultants are obtained by adding the corrections to the previous values

$$r_i^{0[n+1]} = r_i^{0[n]} + \Delta r_i^0 \tag{103}$$

$$r_i^{L[n+1]} = r_i^{L[n]} + \Delta r_i^L \tag{104}$$

$$a_i^{0[n+1]} = a_i^{0[n]} + \Delta a_i^0 \tag{105}$$

$$b_i^{0[n+1]} = b_i^{0[n]} + \Delta b_i^0. \tag{106}$$

The other variables are, however, not additive. $\Delta\vartheta_i^0$ and $\Delta\vartheta_i^L$ are incremental rotational vectors at boundary points. Because the rotations are the multiplicative group, the corresponding boundary rotation matrices are obtained by the multiplications

$$\mathrm{R}_{ij}^{0[n+1]} = \Delta\mathrm{R}_{ik}^0\mathrm{R}_{kj}^{0[n]} \tag{107}$$

$$\mathrm{R}_{ij}^{L[n+1]} = \Delta\mathrm{R}_{ik}^L\mathrm{R}_{kj}^{L[n]}. \tag{108}$$

The Spurrier algorithm [Spurrier (1978)] is then used to extract new components of the boundary rotational vectors $\vartheta_i^{0[n+1]}$ and $\vartheta_i^{L[n+1]}$ from $\mathrm{R}_{ij}^{0[n+1]}$ and $\mathrm{R}_{ij}^{L[n+1]}$.

The rotation matrix, the rotational vector, and the curvature vector along the axis of the beam are obtained as follows. The corrections of the components $\Delta\kappa_i^*$ at any point of the axis of the beam are obtained using the interpolation (95)

$$\Delta\kappa_i^*(x) = I_p(x)\,\Delta\kappa_i^{*p}. \tag{109}$$

Inserting (109) into (77) and integrating gives the corrections of the components of the rotational vector

$$\Delta\vartheta_i(x) = \Delta\vartheta_i^0 + Q_p(x)\,\Delta\kappa_i^{*p}. \tag{110}$$

With the corrections of the rotational vector evaluated, the incremental rotation matrix is found by employing Eq. (82), and new components of the total current rotation matrix are obtained by the multiplication of the two rotation matrices, $\Delta\mathsf{R}(x)$ and $\mathsf{R}^{[n]}(x)$

$$\mathrm{R}_{ij}^{[n+1]}(x) = \Delta\mathrm{R}_{ik}(x)\,\mathrm{R}_{kj}^{[n]}(x). \tag{111}$$

From the updated total rotation matrix, the total rotational vector is extracted. It is easy to see that the update of the components of the curvature vector can be made using the relationship (78) between κ_i^{*p}, ϑ_i^p, and κ_i^p in the exact incremental form

$$\Delta\kappa_{i,g}(x) = \mathrm{T}_{jk}(\Delta\vartheta_i(x))\,\Delta\kappa_i^*(x). \tag{112}$$

Observe that, for the sake of clarity, the components of the curvature vector have been expressed with respect to the spatial basis. In fact, prior to the curvature update they must be transformed into the material basis in iteration step $n+1$

$$\Delta\kappa_i(x) = \mathrm{R}_{ij}^{[n+1]}(x)\,\Delta\kappa_{i,g}(x). \tag{113}$$

Only then the curvature vectors are additive

$$\kappa_i^{[n+1]}(x) = \kappa_i^{[n]}(x) + \Delta\kappa_i(x). \tag{114}$$

The proof of additivity of the curvature vectors is analogous to the one for the additivity of the angular velocity vectors. See, e.g., Shabana (1998).

Remark 8 *The update procedure proposed above is different from procedures employed by other formulations. It is crucial for the objectivity of the discrete strain measures, and for the path independent results in conservative systems.*

6 Numerical examples

In this section, we consider several numerical examples to demonstrate the performance and accuracy of the proposed formulation. For comparison with other formulations, in all numerical examples a linear elastic material is employed in which the operators C_N and C_M in (20)–(21) are taken to be diagonal matrices

$$\boldsymbol{N} = \begin{bmatrix} EA_1 & 0 & 0 \\ 0 & GA_2 & 0 \\ 0 & 0 & GA_3 \end{bmatrix} (\boldsymbol{\gamma} - \boldsymbol{\gamma}_0)$$

$$\boldsymbol{M} = \begin{bmatrix} GJ_1 & 0 & 0 \\ 0 & EJ_2 & 0 \\ 0 & 0 & EJ_3 \end{bmatrix} (\boldsymbol{\kappa} - \boldsymbol{\kappa}_0).$$

Here E and G denote elastic and shear moduli of material; A_1 is the cross-sectional area, J_1 is the torsional inertial moment of the cross-section; A_2 and A_3 are the shear areas in the principal directions 2 and 3 of the cross-section; J_2 and J_3 are the cross-sectional inertial moments about the principal axes 2 and 3.

Different types of elements are used in order to investigate the influence of the number of interpolation (and global integration) nodes and the order of the local integration on the accuracy of numerical solutions. To distinguish between elements, each element is marked by the symbol 'E' and equipped by two indices, E_{N-M}; N is the number of the interpolation nodes, and M is the number of additional internal points between the two subsequent global integration nodes used for the local integration. Recall that the incremental curvatures are interpolated by the polynomials of degree $N-1$. Each element has thus $18+3N$ degrees of freedom (see Section 5.2). The storage of local $3MN$ values of the rotational vector components is needed as explained by the update procedure presented in Section 5.4.

The quadratic convergence of Newton's method was achieved in all numerical examples. The iteration was terminated when the Euclidean norm of the vector of nodal unknowns, $\|\Delta\boldsymbol{y}\|_2$, and of the vector of unbalanced forces, $\|\boldsymbol{h}\|_2$, was less than 10^{-13}.

Based on the present finite element formulation, a special computer code was written for Matlab, release 11 for Windows [The MathWorks (1999)]. Numerical examples were run on an Intel based PC.

6.1 Illustration of objectivity

The present example shows – by the way of numerical testing – the objectivity of the discrete strain measures of the present formulation. We assume a given deformed configuration of a single element with the components of the curvature κ_i known at 30 local points of the axis in addition to the global nodes of the element (see Fig. 4)

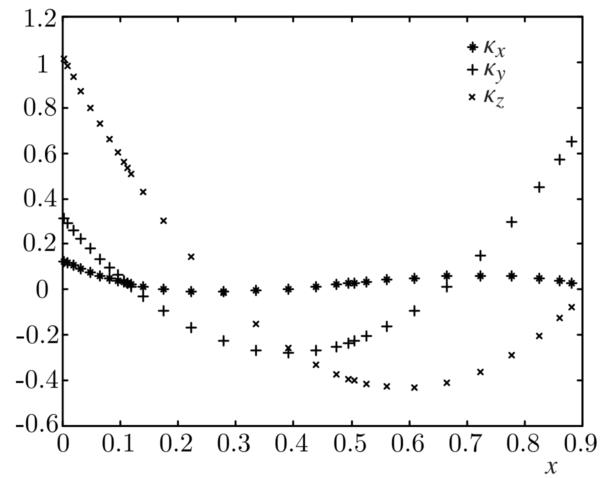

Figure 4 : Curvatures of the axis of an element at local poins.

Then a variety of rigid-body rotations $\mathbf{R}_R$ was superimposed. They were parametrized by a randomly chosen rotational vector $\boldsymbol{\vartheta}_R$. From the prescribed rigid rotational vector the corresponding displacements and rotations of the boundary nodes of the element were calculated and inserted into the right-hand side of the iteration scheme (102). Using the tangent stiffness matrix of the previous deformed configuration and new value of the right-hand side vector, the current composed configuration was obtained in one iteration step, describing the consequences of the superimposed rotation. New values of the components of the curvature vector at previously chosen local points were obtained and compared with their previous values.

Because the strain measures are invariant under a superimposed rigid-body motion, new values should be equal to the old ones. Only very small differences are allowed due to the finite arithmetic of the computer. We used an Intel based PC with a floating point arithmetic of the machine precision of order $2 \cdot 10^{-16}$. In the numerical tests, the absolute error of all components of the curvature vec-

$\|\boldsymbol{\vartheta}_R\|_2$	$\boldsymbol{\vartheta}_R^T$	max. absolute error in κ_i
1.08	$[0.8, 0.7, 0.2]$	$1.11 \cdot 10^{-16}$
3.92	$[1.8, 2.7, 2.2]$	$1.11 \cdot 10^{-16}$
9.75	$[4.8, 6.7, 5.2]$	$1.66 \cdot 10^{-16}$
11.29	$[4.8, 9.7, 3.2]$	$5.55 \cdot 10^{-17}$

Table 1 : The test of objectivity of strain measures

tor were calculated. Very small differences were found indeed. Therefore, only the maximum of all absolute errors of all the 90 components is stated. In Tab. 1 we present the results for four different rigid rotational vectors. Each of the results show that the order of maximum absolute error of the curvatures is less than the machine precision ($2 \cdot 10^{-16}$), irrespective of the size of the rigid rotation.

6.2 Illustration of path independence

In order to study the path independence, a cantilever under a large point force at its free end was considered. The force was applied to the cantilever in different number of loading steps, $\lambda = 1$, 2, 10, 50, and 100. The absolute errors of all mechanical variables (including the stress resultants) with respect to the results at $\lambda = 1$ were calculated. Only very small differences (less then 10^{-14}) between the results using different number of loading steps were found (see Tab. 2). This indicates the path independence of the present formulation.

λ	max. absolute error of results
2	10^{-15}
10	10^{-15}
50	10^{-15}
100	10^{-14}

Table 2 : The test of path independence

6.3 Lateral buckling of a cantilever

We consider a straight, inextensible, shear and in-plane bending stiff cantilever, subjected to the point force at its free end (see Fig. 5). The lateral out-of-plane buckling load F_c is sought. The numerical results are compared with the analytical solution provided by Timoshenko and Gere (1961).

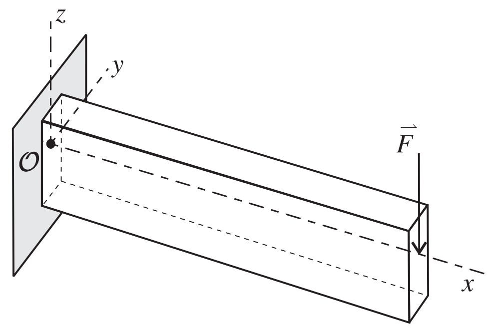

Figure 5 : Lateral buckling of a cantilever.

The numerical solution is obtained iteratively employing the condition that the critical load represents the load at which the tangent stiffness matrix becomes singular. The inextensibility, shear and in-plane bending rigidity were approximated by taking large values for GA_y, GA_z, EA, and EJ_y (Fig. 5). The selected properties of the cantilever were

$$
\begin{aligned}
GA_y = GA_z = EA = EJ_y &= 10^{15} \\
EJ_z &= 1250 \\
GJ_t &= 50 \\
L &= 100.
\end{aligned}
$$

In Tab. 3 the influence of the number of elements, the number of interpolation nodes of an element, and of the number of local integration points on the critical load is displayed. Higher accuracy of the numerical solution is obtained by increasing the number of elements and/or by increasing the number of interpolation nodes of an element. The influence of the order of the local integration is also displayed, although its influence is only minor.

When employing a single element with two nodes, a rather substantial error is found. Increasing the number of two-noded elements, yields a more accurate result. To obtain a nine-digit accurate solution, 20 elements with 3 nodes are required; such a mesh has 540 degrees of freedom. Equally accurate results are obtained by using one 8-noded element with 42 degrees of freedom. We should emphasize the similarity of our results to the results of the element proposed by Jelenić and Saje (1995). The results fully agree, the only difference being that the formulation by Jelenić and Saje requires the interpolation polynomial of one degree higher compared to the present formulation.

e.t.	d.o.f.	$n_e = 1$	$n_e = 2$	$n_e = 5$	$n_e = 10$	$n_e = 20$
E_{2-2}		0.112219000	0.101432352	0.100349434	0.100317209	0.100315118
E_{2-10}		0.112219000	0.101432352	0.100349434	0.100317209	0.100315118
E_{3-2}	540	0.101375990	0.100349169	0.100315163	0.100314987	0.100314984
E_{3-3}	540	0.101375990	0.100349169	0.100315163	0.100314987	0.100314984
E_{3-10}	540	0.101375990	0.100349169	0.100315163	0.100314987	0.100314984
E_{4-2}	300	0.100405493	0.100315809	0.100314980	0.100314984	
E_{4-4}	300	0.100406687	0.100315886	0.100314980	0.100314984	
E_{4-10}	300	0.100406687	0.100315886	0.100314980	0.100314984	
E_{5-2}		0.100320170	0.100314984	0.100314980		
E_{5-5}		0.100320935	0.100315007	0.100314980		
E_{5-10}		0.100320935	0.100315007	0.100314980		
E_{6-2}	180	0.100315089	0.100314970	0.100314980		
E_{6-6}	180	0.100315404	0.100314984	0.100314984		
E_{6-10}	180	0.100315404	0.100314984	0.100314984		
E_{7-2}	78	0.100314839	0.100314981			
E_{7-7}	78	0.100315000	0.100314984			
E_{7-10}	78	0.100315000	0.100314984			
E_{8-2}	42	0.100314980				
E_{8-5}	42	0.100314983				
E_{8-8}	42	0.100314983				
analytical solution		**0.100314984**				

e.t.=element type, n_e=number of elements

Table 3 : The out-of-plane buckling load.

6.4 *Cantilever bent to a helical form*

We consider a very interesting example, presented by Ibrahimbegovic (1997). When a straight in-plane cantilever is subjected to a point moment at its free end, it deforms into a part of a circle which results in a pure bending of the cantilever. A much more interesting behavior is observed when an out-of-plane point force is added at the free end of the cantilever. The out-of-plane force causes displacements of a beam in the out-of-plane direction. When the two point loads are applied simultaneously, the beam bends into the helical form.

We took the same geometric and material properties of the cantilever as in Ibrahimbegovic (1997) :

$$
\begin{aligned}
GA_y = GA_z = EA &= 10^4 \\
EJ_y = EJ_z = GJ_t &= 10^2 \\
L &= 10 \\
M &= 200\pi\lambda \\
F &= 50\lambda.
\end{aligned}
$$

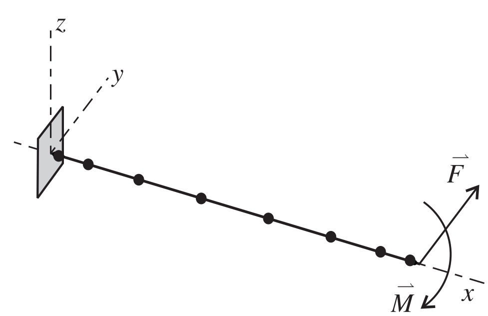

Figure 6 : Ilustration of a one-element, 8-noded model of cantilever subjected to the point load and moment at its free end.

The two loads increase incrementally. We used values from $\lambda = 0$ to $\lambda = 1$ with the step 0.001. The beam was modelled by a mesh of 5 elements with 8 interpolation nodes. The nodes chosen are the points of the Gaussian integration on the interval $[0, L]$, as illustrated in Fig. 6, and are therefore not equally spaced. This choice resulted in a higher accuracy of integration.

The displacements of the free end of the cantilever are

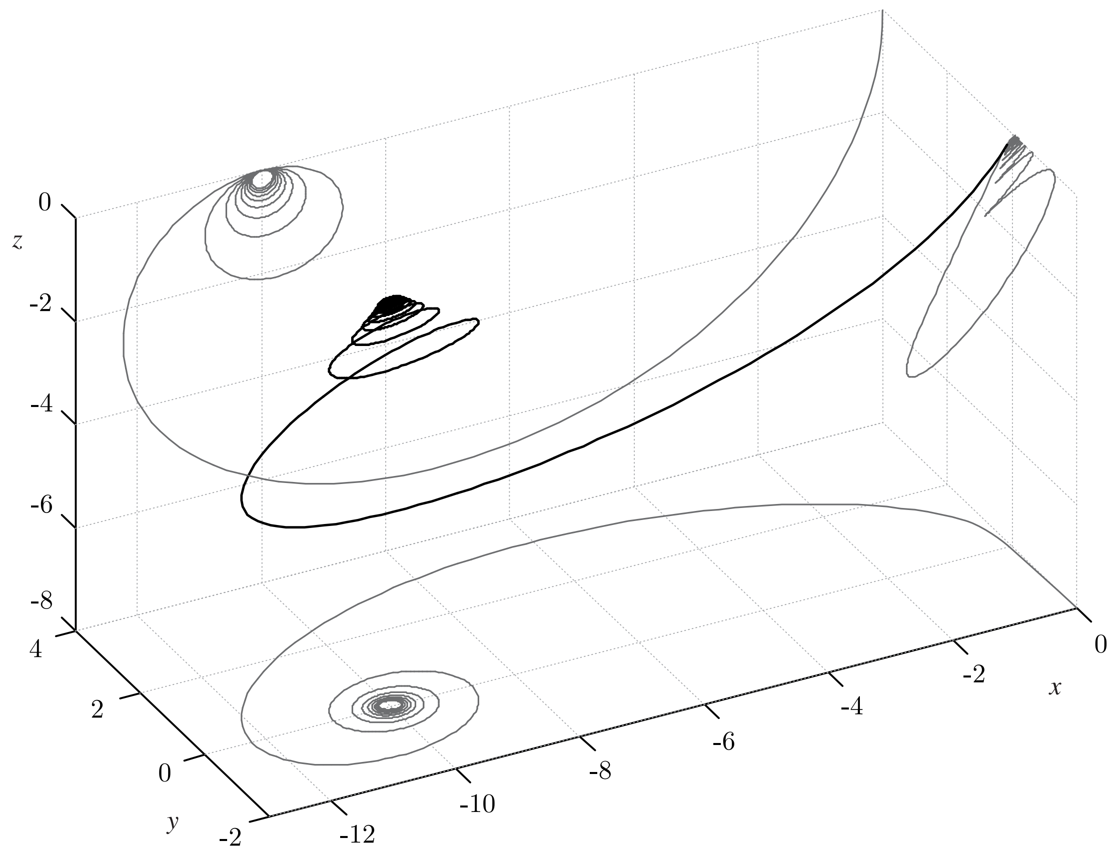

Figure 7 : Displacements of the free end of the cantilever and their projections on the coordinate planes of the spatial coordinate system.

shown in Fig. 7. The projections on the planes ('shadows') are added to give a reader better impression. The comparison of the values of the components of the free-end displacements in the direction of the applied force shows almost complete agreement with the results by Ibrahimbegovic (Fig. 8).

7 Conclusions

A new finite element formulation of the kinematically exact three-dimensional beam theory based on the interpolation of curvature is presented. The essential characteristics of the formulation are:

(i) A modified principle of virtual work is proposed in which the only independent unknown function is the variation of the curvature vector.

(ii) Hence it follows that the only function that needs to be interpolated is the iterative increment (or the variation) of the curvature vector $\delta\boldsymbol{\kappa}^*$; this vector represents the energy complement to the moment vector $\boldsymbol{M}$ given with respect to the spatial basis. The number of interpolation points marks the order of the finite element.

(iii) Displacements and rotational vectors (or their variations) are not interpolated.

(iv) The consistency condition that the equilibrium and the constitutive internal force and moment vectors are equal, is enforced to be satisfied at the interpolation points (the 'collocation'). This considerably improves the accuracy of the calculated internal forces and moments in materially non-linear (e.g., visco-plastic) problems. The abscisae of Gaussian integration points were here employed as the interpolation points although the choice is arbitrary.

(v) The determination of internal forces and moments does not require the differentiation. It then follows that the accuracy of the internal forces and moments

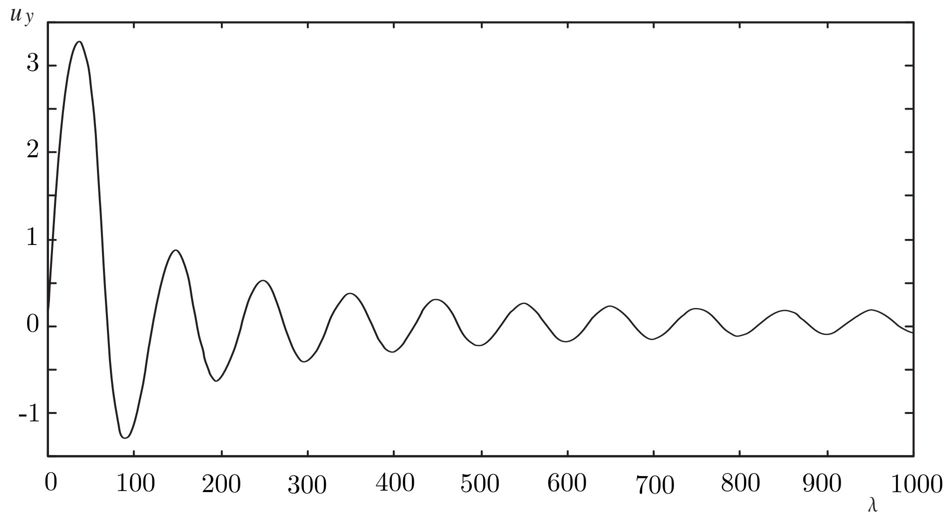

Figure 8 : Displacements of the free end of the cantilever in the direction of the applied force.

is of the same order as the accuracy of the basic variable – the curvature. This is an important advantage compared to formulations where the derivatives are needed for the evaluation of internal forces.

(vi) The matrices of a finite element are derived with respect to the global coordinate system. The coordinate transformation from the local to the global system is thus not necessary. An arbitrary initial curvature and deformation of the beam can be assumed at the initial unloaded configuration.

(vii) The objectivity of the discrete strain measures, i.e., their invariance to rigid-body motions, as implemented in the present finite element scheme, was confirmed by numerical examples.

(viii) The path independence for conservative loadings was also confirmed by numerical examples.

(ix) The present finite elements are free of locking.

(x) A number of different-order finite elements have been tested by various numerical examples. A rapid convergence of results with increasing number of elements has been the characteristic of the formulation.

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Appendix A: Derivation of equations (99)

Partial derivatives are obtained using the connection between the variation of a scalar function and the variation of its variables

$$\delta f\left(x_1,\ldots,x_n\right)=\frac{\partial f}{\partial x_1}\delta x_1+\ldots+\frac{\partial f}{\partial x_n}\delta x_n. \qquad (115)$$

By inserting the interpolation (95) into (14) and integrating the equation, we obtain

$$\begin{aligned}\delta\vartheta_i\left(x\right)&=\delta\vartheta_i^0+\int_0^x I_p\left(\xi\right)\delta\kappa_i^{*p}d\xi\\&=\delta\vartheta_i^0+Q_p\left(x\right)\delta\kappa_i^{*p}. \qquad (116)\end{aligned}$$

From (115) it follows

$$\frac{\partial\vartheta_i}{\partial\vartheta_i^0}=1$$
$$\frac{\partial\vartheta_i}{\partial\kappa_i^{*p}}=Q_p$$

while all the other partial derivatives of ϑ_i vanish. By rewriting the variation of the rotation matrix $\delta\mathbf{R}=\delta\mathbf{\Theta}\mathbf{R}$ into the component form

$$\delta \mathrm{R}_{ij}=-e_{ikm}\delta\vartheta_m\mathrm{R}_{kj} \qquad (117)$$

and inserting (116) into (117), we obtain the partial derivatives of the components of the rotation matrix

$$\frac{\partial \mathrm{R}_{ij}}{\partial\vartheta_m^0}=-e_{ikm}\mathrm{R}_{kj}$$
$$\frac{\partial \mathrm{R}_{ij}}{\partial\kappa_m^{*p}}=-e_{ikm}Q_p\mathrm{R}_{kj}.$$

From (12) and (95) the variations of κ_i are expressed by

$$\delta\kappa_i=\mathrm{R}_{ji}I_p\delta\kappa_j^{*p},$$

which leads to

$$\frac{\partial\kappa_i}{\partial\kappa_j^{*p}}=\mathrm{R}_{ji}I_p.$$

The partial derivatives of γ_i are obtained by varying Eq. (79) in conjunction with (80)

$$C_i^N-\mathrm{R}_{ji}a_j=0.$$

Using (100) gives

$$\mathrm{C}_{ik}\delta\gamma_k+\mathrm{C}_{i,r+3}\delta\kappa_r-\delta\mathrm{R}_{ji}a_j-\mathrm{R}_{ji}\delta a_j=0. \qquad (118)$$

Inserting (116) and (117) into (118), and taking into account that $\delta a_j=\delta a_j^0$ in the case of conservative loads, yields

$$\begin{aligned}&\mathrm{C}_{ik}\delta\gamma_k+\mathrm{C}_{i,r+3}\mathrm{R}_{mr}I_p\delta\kappa_m^{*p}\\&\qquad+e_{jnm}\left(\delta\vartheta_m^0+Q_p\delta\kappa_m^{*p}\right)\mathrm{R}_{ni}a_j-\mathrm{R}_{ji}\delta a_j^0=0.\end{aligned}$$

By rearranging terms, the variation of γ_k is expressed by the variations of the basic unknown variables a_j^0, ϑ_m^0, and κ_m^{*p} as

$$\begin{aligned}\mathrm{C}_{ik}\delta\gamma_k&=\mathrm{R}_{ji}\delta a_j^0-e_{jnm}\mathrm{R}_{ni}a_j\delta\vartheta_m^0\\&\quad-\left(\mathrm{C}_{i,r+3}\mathrm{R}_{mr}I_p+e_{jnm}Q_p\mathrm{R}_{ni}a_j\right)\delta\kappa_m^{*p}.\end{aligned}$$

Hence, the partial derivatives of γ_k are

$$\frac{\partial\gamma_k}{\partial a_j^0}=\tilde{\mathrm{C}}_{ki}\mathrm{R}_{ji}$$
$$\frac{\partial\gamma_k}{\partial\vartheta_m^0}=-\tilde{\mathrm{C}}_{ki}e_{jnm}\mathrm{R}_{ni}a_j$$
$$\frac{\partial\gamma_k}{\partial\kappa_m^{*p}}=-\tilde{\mathrm{C}}_{ki}\mathrm{C}_{i,r+3}\mathrm{R}_{mr}I_p-\tilde{\mathrm{C}}_{ki}e_{jnm}Q_p\mathrm{R}_{ni}a_j.$$

Here, $\tilde{\mathrm{C}}_{ki}$ denotes the components of the inverse matrix of the 3×3 upper left-corner sub-matrix of material tangent matrix $\mathbf{C}$:

$$\begin{bmatrix}\mathrm{C}_{11}&\mathrm{C}_{12}&\mathrm{C}_{13}\\\mathrm{C}_{21}&\mathrm{C}_{22}&\mathrm{C}_{23}\\\mathrm{C}_{31}&\mathrm{C}_{32}&\mathrm{C}_{33}\end{bmatrix}^{-1}=\mathbf{C}_{\gamma\gamma}^{-1}=\tilde{\mathbf{C}}_{\gamma\gamma}.$$