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On Stress Resultant Plasticity and Viscoplasticity for Metal Plates

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Abstract

In this work we derive elastoplastic and elastoviscoplastic finite element formulations for stress resultant bending analysis of thin metal plates. The principle of maximum plastic dissipation is used to obtain the ingredients of the small strain stress resultant plate elastoplasticy with state variables describing general isotropic and linear kinematic hardening. The ingredients of the plate stress resultant elastoviscoplasticy are further obtained by using the penalty-like form of the principle of maximum plastic dissipation. Such an approach enables single framework for numerical implementation of both considered inelastic stress resultant plate material models. The implementation is based on the spectral decomposition algorithm. For spatial discretization we use simple and robust quadrilateral finite element. A set of numerical examples is presented to illustrate the approach and to discuss the accuracy of the stress resultant inelastic plate formulations.

Key words: plates, finite elements, stress resultant plasticity, stress resultant viscoplasticity

1 Introduction

There are two ways to derive a computational formulation for elastoplastic bending of thin metal plates. More common approach defines the elastoplastic constitutive equations for plates in terms of stresses; see e.g. Brank et al. [3] for the shell case. The numerical integration of stresses through the plate thickness is then performed to evaluate the stress resultants, i.e. the moments and the transverse shear forces per unit length, in order to solve the 2d plate boundary value problem. Another way, which is less common, introduces the elastoplastic constitutive equations directly in the 2d stress resultant form; see e.g. Simo and Kennedy [22], Skallerud et al. [23], Crisfield and Peng [5] for the shell case.

The latter approach is on one hand computationally much faster than the former one, but on the other hand it fails to describe the spreading of the plastification through the plate thickness, see e.g. Auricchio and Taylor [1]. This drawback can be removed (to a certain extent) by a pseudo-time dependent value of the yield parameter associated with the plate bending response. This was first suggested by Crisfield, see e.g. [4] and references therein, and was later used e.g. by Shi and Voyiadjis [20] for plates and Zeng et al. [25] and Voyiadjis and Woelke [24] for shells. Another way to approximately describe the 3d effect of spreading of plasticity throughout the plate is to use the generalized plasticity model for plates, which is based on two functions (both defined in terms of stress resultants), the yield function and the limit function, see e.g. Auricchio and Taylor [1]. However, if one wants to evaluate only the limit load of the metal plate, the above mentioned modifications are unnecessary, since both the stress resultant formulation and the stress based formulation with the through-the-thickness numerical integration provide the same result.

In this work we derive small strain elastoplastic plate finite element formulation in terms of stress resultants. Nonlinear isotropic and linear kinematic hardening are considered. We further extend the plasticity formulation into the visoplasticy formulation of Perzyna type, e.g. Kojić and Bathe [14], Kleiber and Kowalczyk [13]. Both elastoplastic and elastoviscoplastic stress resultant plate formulations are derived by exploiting the hypotheses of instantaneous elastic response and the principle of maximum plastic dissipation (plasticity) or the penalty-like form of the principle of maximum plastic dissipation (viscoplasticity); see e.g. Ibrahimbegovic et al. [11] for 3d setting of those topics. We show that with such an approach a unified computational framework for elastoplastic and elastoviscoplastic stress resultant plate analysis can be obtained.

The paper is organized as follows: In section 2 we systematically derive basic equations of elastoplastic and elastoviscoplastic plate models. In section 3 we present spatial finite element discretization and numerical procedure for integration of elastoplastic and elastoviscoplastic evolution equations. The finite element that is used is the simplest of the family of the plate elements presented in Bohinc et al. [2] and Ibrahimbegovic [9], which share the property that the interpolation of transverse displacement is one order higher than the interpolation of rotations. Numerical examples are presented in section 4 and concluding remarks are drawn in section 5.

2 Inelastic plate models

We model a plate as a 2d body occupying a domain Ω in the x_1x_2 plane. The weak form of the equilibrium equations is for the inelastic geometrically linear Reissner-Mindlin plate model given as

$$G\left(m_{\alpha\beta}, q_\alpha; \widehat{w}, \widehat{\theta}_\gamma\right) = \int_\Omega \widehat{\kappa}_{\alpha\beta}\left(\widehat{\theta}_\gamma\right) m_{\alpha\beta} d\Omega + \int_\Omega \widehat{\gamma}_\alpha \left(\widehat{w}, \widehat{\theta}_\gamma\right) q_\alpha d\Omega, \tag{1}$$

$$-\int_\Omega \widehat{w} p d\Omega - G_{ext,b}\left(\overline{m}_\alpha, \overline{q}_\alpha; \widehat{w}, \widehat{\theta}_\gamma\right) = 0, \qquad \alpha, \beta, \gamma \in \{1, 2\},$$

where w is transverse displacement (i.e. displacement in the direction of x_3 coordinate), θ_γ is rotation of a plate normal (i.e. unit vector in the direction of x_3 coordinate) around x_γ axis, $\kappa_{\alpha\beta}$ are bending strains, γ_α are shear strains

$$\kappa_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial \beta_\alpha}{\partial x_\beta} + \frac{\partial \beta_\beta}{\partial x_\alpha}\right), \qquad \beta_\alpha = e_{\alpha\beta}\theta_\beta \Rightarrow \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \tag{2}$$

$$\gamma_\alpha = \frac{\partial w}{\partial x_\alpha} - \beta_\alpha, \tag{3}$$

$m_{\alpha\beta}$ are bending moments, q_α are transverse forces, p is transverse plate loading (i.e. loading in the direction of x_3 coordinate), $G_{ext,b}$ is virtual work of external moments $\overline{m}_\alpha$ and external forces $\overline{q}_\alpha$ acting on the plate boundary, and $\widehat{(\circ)}$ is virtual quantity that corresponds to $(\circ)$. We consider displacement w, rotations θ_γ, stress resultants $m_{\alpha\beta}$, q_α and load as functions of position $\mathbf{x} = [x_1, x_2]^T \in \Omega$ and pseudo-time $t \in [0, T]$, i.e. $w = w(\mathbf{x}, t)$, $\theta_\gamma = \theta_\gamma(\mathbf{x}, t)$, $m_{\alpha\beta} = m_{\alpha\beta}(\mathbf{x}, t)$, $q_\alpha = q_\alpha(\mathbf{x}, t)$, $p = p(\mathbf{x}, t)$, $\overline{m}_\alpha = \overline{m}_\alpha(\mathbf{x}, t)$, $\overline{q}_\alpha = \overline{q}_\alpha(\mathbf{x}, t)$. Equation (1) can be written in matrix form as

$$G\left(\mathbf{m}, \mathbf{q}; \widehat{w}, \widehat{\boldsymbol{\theta}}\right) = \int_\Omega \widehat{\boldsymbol{\kappa}}^T \mathbf{m} d\Omega + \int_\Omega \widehat{\boldsymbol{\gamma}}^T \mathbf{q} d\Omega - \int_\Omega \widehat{w} p d\Omega - G_{ext,b} = 0, \tag{4}$$

where the following mappings have been defined

$$\begin{aligned}
\theta_\gamma &\mapsto \boldsymbol{\theta} = [\theta_1, \theta_2]^T, \\
\kappa_{\alpha\beta} &\mapsto \boldsymbol{\kappa} = [\kappa_{11}, \kappa_{22}, 2\kappa_{12}]^T = \left[-\frac{\partial \theta_2}{\partial x_1}, \frac{\partial \theta_1}{\partial x_2}, \frac{\partial \theta_1}{\partial x_1} - \frac{\partial \theta_2}{\partial x_2}\right]^T, \\
\gamma_\alpha &\mapsto \boldsymbol{\gamma} = [\gamma_1, \gamma_2]^T = \left[\frac{\partial w}{\partial x_1} + \theta_2, \frac{\partial w}{\partial x_2} - \theta_1\right]^T, \\
m_{\alpha\beta} &\mapsto \mathbf{m} = [m_{11}, m_{22}, m_{12}]^T, \qquad q_\alpha \mapsto \mathbf{q} = [q_1, q_2]^T.
\end{aligned} \tag{5}$$

For further use we also define the following strain and stress resultant vectors

$$\begin{aligned}
\boldsymbol{\varepsilon} &= \left[\boldsymbol{\kappa}^T, \boldsymbol{\gamma}^T\right]^T = [\kappa_{11}, \kappa_{22}, 2\kappa_{12}, \gamma_1, \gamma_2]^T, \\
\boldsymbol{\sigma} &= \left[\mathbf{m}^T, \mathbf{q}^T\right]^T = [m_{11}, m_{22}, m_{12}, q_1, q_2]^T.
\end{aligned} \tag{6}$$

Having defined the weak form of equilibrium equations and the kinematic relations, we proceed with stress resultant inelastic constitutive relations for small strain plate bending problems.

2.1 Plate elastoplasticity

We consider the following internal variables to describe the irreversible nature of the plastic process during the plate bending: the plastic strain $\boldsymbol{\varepsilon}^p$, the scalar parameter ξ, which controls the isotropic hardening mechanism, and the strain-like parameters $\varkappa_{ij}$, $i,j \in \{1,2,3\}$, which control the kinematic hardening mechanism. The state variables are functions of position $\mathbf{x}$ and pseudo-time t, i.e. $\boldsymbol{\varepsilon}^p = \boldsymbol{\varepsilon}^p(\mathbf{x},t)$, $\xi = \xi(\mathbf{x},t)$ and $\varkappa_{ij} = \varkappa_{ij}(\mathbf{x},t)$.

A usual additive split of reversible (elastic) and irreversible (plastic) strains is assumed

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p, \tag{7}$$

where, according to (6), $\boldsymbol{\varepsilon}^e = \left[\boldsymbol{\kappa}^{e,T}, \boldsymbol{\gamma}^{e,T}\right]^T$ and $\boldsymbol{\varepsilon}^p = \left[\boldsymbol{\kappa}^{p,T}, \boldsymbol{\gamma}^{p,T}\right]^T$. The strain energy function is assumed to be of the following (quadratic) form

$$\psi\left(\boldsymbol{\varepsilon}^e, \xi, \breve{\varkappa}\right) = \frac{1}{2}\boldsymbol{\varepsilon}^{e,T}\mathbf{C}\boldsymbol{\varepsilon}^e + \Xi(\xi) + \frac{1}{2}\left(\frac{2}{3}H_{kin}\right)\breve{\varkappa}^T\mathbf{D}\breve{\varkappa}, \tag{8}$$

where the mapping $\varkappa_{ij} \mapsto \breve{\varkappa} = [\varkappa_{11}, \varkappa_{22}, 2\varkappa_{12}, 2\varkappa_{13}, 2\varkappa_{23}]^T$ has been defined, and the following matrices have been introduced (we assume the isotropic elastic response of a plate)

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}^b & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^s \end{bmatrix}, \quad \mathbf{C}^b = k^b \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}, \quad \mathbf{C}^s = k^s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{9}$$

Matrix $\mathbf{D}$ in (8) is such that $\breve{\varkappa}^T\mathbf{D}\breve{\varkappa} = \varkappa_{ij}\varkappa_{ij}$, $ij \neq 33$, i.e.

$$\mathbf{D} = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\mathbf{I}_3 \end{bmatrix}, \quad \mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{10}$$

In (8) we assumed a general (nonlinear) form of isotropic hardening and a linear form of kinematic hardening with hardening modulus H_{kin}. The constants in (9) are $k^b = \frac{Eh^3}{12(1-\nu^2)}$, $k^s = \frac{cEh}{2(1+\nu)}$, where E is elastic modulus, ν is Poisson's ratio, h is plate thickness, and c is shear correction factor, usually set to 5/6.

We denote the stress-like internal variables, which correspond to the strain-like internal variables ξ and $\varkappa_{ij}$, as q and α_{ij}, respectively. These dual variables are used to define yield function. In this work we use a stress resultant approximation of the von Mises yield function, which can be for the Reissner-Mindlin plate model written in a non-dimensional form as

$$\phi\left(\boldsymbol{\sigma}, q, \boldsymbol{\alpha}\right) = \left(\boldsymbol{\sigma} + \boldsymbol{\alpha}\right)^T \mathbf{A}\left(\boldsymbol{\sigma} + \boldsymbol{\alpha}\right) - \left(1 - \frac{q}{\sigma_y}\right)^2 = 0, \tag{11}$$

where $\boldsymbol{\sigma}$ is defined in (6), $\boldsymbol{\alpha}$ (the negative of the back stress resultants) is defined by mapping $\alpha_{ij} \mapsto \boldsymbol{\alpha} = [\alpha_{11}, \alpha_{22}, \alpha_{12}, \alpha_{13}, \alpha_{23}]^T$, σ_y is uniaxial yield stress, matrix $\mathbf{A}$ is for isotropic plastic response equal to

$$\mathbf{A} = \begin{bmatrix} \frac{1}{m_0^2}\mathbf{P} & \mathbf{0} \\ \mathbf{0} & \frac{1}{q_0^2}\mathbf{I}_2 \end{bmatrix}, \qquad \mathbf{P} = \frac{1}{2}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \tag{12}$$

and m_0 and q_0 are yield parameters associated with bending and transverse shear, respectively. They are usually set to the fully plastic uniaxial plate bending moment $m_0 = \frac{\sigma_y h^2}{4}$ and to the fully plastic transverse shear force $q_0 = \frac{\sigma_y h}{\sqrt{3}}$. The yield function (11) is special case of generalized (since it includes hardening) Ilyushin-Shapiro stress resultant yield function for shells; see e.g. [5], [22] and references therein for discussion on stress resultant yield functions for shells. A similar form, but without description of kinematic hardening mechanism and with the choice of linear isotropic hardening, was used in [10], [20].

Remark 1: The yield function (11) does not allow to simulate the spreading of plasticity through the plate thickness. One possibility to take through-the-thickness distribution of plasticity into account, while still using a stress resultant form of the yield function, is to multiply the value of m_0 by a parameter α, such that αm_0 follows an experimental uniaxial moment-plastic curvature relation. Some authors, see e.g. [20], [25], used the proposal of Crisfield [4] who suggested the following form of α

$$\alpha(t) = 1 - \frac{1}{3}\exp\left(-\frac{8}{3}\breve{\kappa}^p(t)\right), \tag{13}$$

where

$$\breve{\kappa}^p(t) = \frac{Eh}{\sqrt{3}\sigma_y}\int_0^t \left[\left(\dot{\kappa}_{11}^p\right)^2 + \left(\dot{\kappa}_{22}^p\right)^2 + \dot{\kappa}_{11}^p\dot{\kappa}_{22}^p + \left(\dot{\kappa}_{12}^p\right)^2/4\right]^{\frac{1}{2}} d\tau \tag{14}$$

plays the role of equivalent plastic curvature. In (14) $(\dot{\circ}) = \frac{\partial(\circ)}{\partial t}$. Note that for $\breve{\kappa}^p = 0$ one has $\alpha = 2/3$, and for $\breve{\kappa}^p \to \infty$ one gets $\alpha \to 1$. $\Box$

Remark 2: The general quadratic form of the yield condition (11) is also suitable to express stress resultant approximation of the anisotropic criterion of Hill. In such a case one needs to define the corresponding form of matrix $\mathbf{A}$, see e.g. [25] for details. $\Box$

Having defined plastic strains, strain energy function and yield function, we proceed with derivation of the remaining ingredients of the stress resultant elastoplasticity for plates. For the isothermal case we can write the following rate of material dissipation

$$\mathcal{D} = \boldsymbol{\sigma}^T\dot{\boldsymbol{\varepsilon}} - \frac{d}{dt}\psi\left(\boldsymbol{\varepsilon}^e, \xi, \breve{\boldsymbol{\varkappa}}\right) = \boldsymbol{\sigma}^T\dot{\boldsymbol{\varepsilon}} - \left(\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}^e}\right)^T\left(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^p\right) - \frac{\partial\psi}{\partial\xi}\dot{\xi} - \left(\frac{\partial\psi}{\partial\breve{\boldsymbol{\varkappa}}}\right)^T\dot{\breve{\boldsymbol{\varkappa}}} \geqslant 0, \tag{15}$$

which is assumed to be non-negative. Note that equation (15) can be derived from the second law of thermodynamics, see e.g. [12], [22]. By assuming that the elastic process is non-dissipative (i.e. the state variables do not change during that process and $\mathcal{D} = 0$) one has

$$\boldsymbol{\sigma} = \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}^e} = \mathbf{C}\boldsymbol{\varepsilon}^e. \tag{16}$$

By further consideration of (15) one can define the dual variables, i.e. the hardening variable q and the variables that control kinematic hardening $\boldsymbol{\alpha}$, as

$$q = -\frac{\partial\psi}{\partial\xi} = -\frac{d\Xi(\xi)}{d\xi} = -\Xi'(\xi), \qquad \boldsymbol{\alpha} = -\frac{\partial\psi}{\partial\breve{\boldsymbol{\varkappa}}} = -\frac{2}{3}H_{kin}\mathbf{D}\breve{\boldsymbol{\varkappa}} = -\frac{2}{3}H_{kin}\boldsymbol{\varkappa}, \tag{17}$$

where $\boldsymbol{\varkappa} = [\varkappa_{11}, \varkappa_{22}, \varkappa_{12}, \varkappa_{13}, \varkappa_{23}]^T$. By using (16) and (17) in (15) we obtain the reduced material dissipation (i.e. the dissipation of the plastic process) as

$$\mathcal{D}^p = \boldsymbol{\sigma}^T\dot{\boldsymbol{\varepsilon}}^p + q\dot{\xi} + \boldsymbol{\alpha}^T\dot{\breve{\boldsymbol{\varkappa}}} \geqslant 0. \tag{18}$$

The principle of maximum plastic dissipation states that among all the dual variables $(\boldsymbol{\sigma}, q, \boldsymbol{\alpha})$ that satisfy the yield criteria, one should choose those that maximize plastic dissipation. The problem can be written in the following form: Find minimum of $\mathcal{L}^p(\boldsymbol{\sigma}, q, \boldsymbol{\alpha}, \dot{\gamma})$, where

$$\mathcal{L}^p(\boldsymbol{\sigma}, q, \boldsymbol{\alpha}, \dot{\gamma}) = -\mathcal{D}^p(\boldsymbol{\sigma}, q, \boldsymbol{\alpha}) + \dot{\gamma}\phi(\boldsymbol{\sigma}, q, \boldsymbol{\alpha}), \tag{19}$$

and $\dot{\gamma} \geqslant 0$ plays the role of Lagrange multiplier. From the above minimization problem and (11) we obtain explicit forms of evolution equations for the internal variables

$$\begin{aligned}
\frac{\partial\mathcal{L}^p}{\partial\boldsymbol{\sigma}} &= -\dot{\boldsymbol{\varepsilon}}^p + \dot{\gamma}\frac{\partial\phi}{\partial\boldsymbol{\sigma}} = 0 \Longrightarrow \dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\underbrace{2\mathbf{A}(\boldsymbol{\sigma} + \boldsymbol{\alpha})}_{\boldsymbol{\nu}}, \\
\frac{\partial\mathcal{L}^p}{\partial q} &= -\dot{\xi} + \dot{\gamma}\frac{\partial\phi}{\partial q} = 0 \Longrightarrow \dot{\xi} = \dot{\gamma}\underbrace{\frac{2}{\sigma_y}\left(1 - \frac{q}{\sigma_y}\right)}_{\beta} \overset{(11)}{=} \dot{\gamma}\underbrace{\frac{2}{\sigma_y}\sqrt{(\boldsymbol{\sigma} + \boldsymbol{\alpha})^T\mathbf{A}(\boldsymbol{\sigma} + \boldsymbol{\alpha})}}_{\beta}, \\
\frac{\partial\mathcal{L}^p}{\partial\boldsymbol{\alpha}} &= -\dot{\breve{\boldsymbol{\varkappa}}} + \dot{\gamma}\frac{\partial\phi}{\partial\boldsymbol{\alpha}} = 0 \Longrightarrow \dot{\breve{\boldsymbol{\varkappa}}} = \dot{\gamma}\underbrace{2\mathbf{A}(\boldsymbol{\sigma} + \boldsymbol{\alpha})}_{\boldsymbol{\nu}}.
\end{aligned} \tag{20}$$

Note that $\dot{\varkappa} = \dot{\varepsilon}^p$.

Remark 3: Equation $(20)_2$ is generalization of the equivalent plastic work variable $W^p = \boldsymbol{\sigma}^T \dot{\boldsymbol{\varepsilon}}^p$. Namely, by inserting $\dot{\varkappa} = \dot{\varepsilon}^p$ and $(20)_1$ into (18), and using (11), one gets $\mathcal{D}^p = \dot{\gamma} 2 \left(1 - \frac{q}{\sigma_y}\right)$, which implies, see $(20)_2$, that $\dot{\xi} = \frac{1}{\sigma_y} \mathcal{D}^p$. $\square$

The loading/unloading conditions follow from the demands that $\dot{\gamma}$ is non-negative, ϕ is non-positive, and the plastic dissipation $\mathcal{D}^p$ equals zero for elastic process when $\phi < 0$

$$\dot{\gamma} \geqslant 0, \qquad \phi \leqslant 0, \qquad \dot{\gamma}\phi = 0. \tag{21}$$

In addition to (21) we have the condition $\dot{\phi} = 0$ if $\dot{\gamma} > 0$ (the consistency condition). It guarantees the admissibility of the subsequent state in the case of change of state variables. The consistency condition

$$\dot{\gamma} > 0; \quad \dot{\phi} = 0 = \left(\frac{\partial \phi}{\partial \boldsymbol{\sigma}}\right)^T \dot{\boldsymbol{\sigma}} + \frac{\partial \phi}{\partial q} \dot{q} + \left(\frac{\partial \phi}{\partial \boldsymbol{\alpha}}\right)^T \dot{\boldsymbol{\alpha}}, \tag{22}$$

pseudo-time derivatives of (16) and (17), $\dot{\boldsymbol{\sigma}} = \mathbf{C}\left(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^p\right)$, $\dot{q} = -\Xi''(\xi)\dot{\xi}$, $\dot{\boldsymbol{\alpha}} = -\frac{2}{3} H_{kin} \dot{\boldsymbol{\varkappa}}$, and equations (20) lead to the following expression for $\dot{\gamma}$

$$\dot{\gamma} = \frac{1}{\left(\boldsymbol{\nu}^T \mathbf{C} \boldsymbol{\nu} + \Xi''(\xi)\beta^2 + \frac{2}{3} H_{kin} \boldsymbol{\nu}^T \boldsymbol{\nu}\right)} \boldsymbol{\nu}^T \mathbf{C} \dot{\boldsymbol{\varepsilon}}. \tag{23}$$

If (23) and (20) are used in $\dot{\boldsymbol{\sigma}} = \mathbf{C}\left(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^p\right)$, one can write $\dot{\boldsymbol{\sigma}} = \mathbf{C}^{ep} \dot{\boldsymbol{\varepsilon}}$, where

$$\mathbf{C}^{ep} = \left\{ \begin{array}{cl} \mathbf{C} & \text{if } \dot{\gamma} = 0 \\ \mathbf{C} - \frac{\mathbf{C}\boldsymbol{\nu}\boldsymbol{\nu}^T\mathbf{C}}{\boldsymbol{\nu}^T \mathbf{C} \boldsymbol{\nu} + \Xi''(\xi)\beta^2 + \frac{2}{3} H_{kin} \boldsymbol{\nu}^T \boldsymbol{\nu}} & \text{if } \dot{\gamma} > 0 \end{array} \right. \tag{24}$$

is elastoplastic tangent modulus of the elastoplastic plate model.

By computation of the internal variables (i.e. by integration of (20)) and by using (16) one recovers the stress resultants $\boldsymbol{\sigma}(\mathbf{x}, t) = \left[\mathbf{m}^T, \mathbf{q}^T\right]^T$ appearing in the weak form of the equilibrium equations (4).

2.2 Plate elastoviscoplasticity

A stress resultant viscoplastic constitutive equations for plates of Perzyna type are obtained by a modification of the elastoplasticity model presented in the previous section. The basic difference between the viscoplasticity and plasticity is that in the former model the stress states $\{\boldsymbol{\sigma}, q, \boldsymbol{\alpha}\}$, such that $\phi(\boldsymbol{\sigma}, q, \boldsymbol{\alpha}) > 0$, are permissible, while in the latter are not. The state variables remain the same, except for the viscoplastic strain $\boldsymbol{\varepsilon}^{vp}$, which replaces $\boldsymbol{\varepsilon}^p$. The constrained minimization problem (19) for plasticity is here replaced by the penalty form of the principle of maximum plastic dissipation (see e.g. [21] section 2.7 and [11] for details), which can be stated as: Find minimum of $\mathcal{L}^{vp}(\boldsymbol{\sigma}, q, \boldsymbol{\alpha})$, where

$$\mathcal{L}^{vp}(\boldsymbol{\sigma}, q, \boldsymbol{\alpha}) = -\mathcal{D}^{vp}(\boldsymbol{\sigma}, q, \boldsymbol{\alpha}) + \frac{1}{\eta} g\left(\phi(\boldsymbol{\sigma}, q, \boldsymbol{\alpha})\right), \tag{25}$$

$\eta \in (0, \infty)$ is penalty parameter (also called viscosity coefficient or fluidity parameter), $\mathcal{D}^{vp}$ is viscoplastic dissipation of the same form as (18) and g is penalized functional. A usual choice for g is

$$g(\phi) = \left\{ \begin{array}{ll} \frac{1}{2}\phi^2 & \text{if } \phi \geq 0 \\ 0 & \text{if } \phi < 0 \end{array} \right. . \tag{26}$$

With this choice for g the minimization of (25) leads to

$$\frac{\partial \mathcal{L}^{vp}}{\partial \boldsymbol{\sigma}} = -\dot{\boldsymbol{\varepsilon}}^{vp} + \frac{1}{\eta}\langle\phi\rangle \frac{\partial \phi}{\partial \boldsymbol{\sigma}} = 0, \quad \frac{\partial \mathcal{L}^{vp}}{\partial q} = -\dot{\xi} + \frac{1}{\eta}\langle\phi\rangle \frac{\partial \phi}{\partial q} = 0, \quad \frac{\partial \mathcal{L}^{vp}}{\partial \boldsymbol{\alpha}} = -\dot{\boldsymbol{\varkappa}} + \frac{1}{\eta}\langle\phi\rangle \frac{\partial \phi}{\partial \boldsymbol{\alpha}} = 0, \tag{27}$$

where $\langle\phi\rangle = g\prime(\phi) = \frac{dg}{d\phi}$. Equations (27) provide the corresponding evolution equations of the state variables for the viscoplastic plate model.

If one defines the viscoplastic multiplier $\dot{\gamma}$ as $\dot{\gamma} = \frac{1}{\eta}\langle\phi\rangle$ then the evolution equations for viscoplastic model can be written as those for elastoplastic model, see (20). Their integration, which leads to stress resultants $\boldsymbol{\sigma}(\mathbf{x}, t) = \left[\mathbf{m}^T, \mathbf{q}^T\right]^T$, can be performed in very similar manner as for plasticity, as shown below.

3 Finite element formulation

3.1 Space discretization

In the finite element solution of the plate bending problem a domain under consideration Ω is discretized by a mesh of finite elements so that $\Omega^h = \bigcup_{e=1}^{nel} \Omega^e$, where nel is number of elements in the mesh. In this work we use one of the quadrilateral plate elements originally introduced in [9]. Its geometry is defined by the bilinear mapping $\boldsymbol{\xi} \mapsto \mathbf{x}^h$ ($\boldsymbol{\xi} \in [-1,1] \times [-1,1]$; $\mathbf{x}^h \in \Omega^e$) with

$$\mathbf{x}^h(\boldsymbol{\xi})\,|_{\Omega^e} = \sum_{a=1}^{4} N_a(\boldsymbol{\xi})\,\mathbf{x}_a, \qquad \mathbf{x}_a = [x_{1a}, x_{2a}]^T, \qquad \boldsymbol{\xi} = \left[\xi^1, \xi^2\right]^T, \tag{28}$$

where $\mathbf{x}_a$ are coordinates of the finite element node a and

$$N_a\left(\xi^1, \xi^2\right) = \frac{1}{4}\left(1 + \xi_a^1 \xi^1\right)\left(1 + \xi_a^2 \xi^2\right), \qquad \begin{array}{c|cccc} a & 1 & 2 & 3 & 4 \\ \hline \xi_a^1 & -1 & 1 & 1 & -1 \\ \hline \xi_a^2 & -1 & -1 & 1 & 1 \end{array}. \tag{29}$$

The subscript h is used to denote the discretely approximated quantities. Interpolation of the rotations is based on bilinear polynomials (29)

$$\begin{pmatrix} \theta_1^h \\ \theta_2^h \end{pmatrix} = \boldsymbol{\theta}^h(\boldsymbol{\xi}, t)\,|_{\Omega^e} = \sum_{a=1}^{4} N_a(\boldsymbol{\xi})\,\boldsymbol{\theta}_a(t), \qquad \boldsymbol{\theta}_a = [\theta_{1a}, \theta_{2a}]^T, \tag{30}$$

while interpolation of the transverse displacement is performed in more elaborated way as

$$w^h(\boldsymbol{\xi}, t)\,|_{\Omega^e} = \sum_{a=1}^{4} N_a(\boldsymbol{\xi})\,w_a(t) + \sum_{E=5}^{8} N_E(\boldsymbol{\xi})\,\frac{l_{JK}}{8}\,\mathbf{n}_{JK}^T\left(\boldsymbol{\theta}_J(t) - \boldsymbol{\theta}_K(t)\right). \tag{31}$$

The second term in (31) is such that the shear distribution along each element edge is constant. In (31) $l_{JK} = \left((x_{1K} - x_{1J})^2 + (x_{2K} - x_{2J})^2\right)^{1/2}$, $\mathbf{n}_{JK} = [\cos\alpha_{JK}, \sin\alpha_{JK}]^T$ (see Figure 1) and

$$\begin{array}{ll} N_E(\boldsymbol{\xi}) = \frac{1}{2}\left(1 - \xi^1\right)^2\left(1 + \xi_J^2 \xi^2\right), & E = 5,7 \\ N_E(\boldsymbol{\xi}) = \frac{1}{2}\left(1 - \xi^2\right)^2\left(1 + \xi_J^1 \xi^1\right), & E = 6,8 \end{array}, \qquad \begin{array}{c|cccc} E & 5 & 6 & 7 & 8 \\ \hline J & 1 & 2 & 3 & 4 \\ \hline K & 2 & 3 & 4 & 1 \end{array}. \tag{32}$$

Interpolation of bending strains follows from $(5)_2$ and (30)

$$\boldsymbol{\kappa}^h(\boldsymbol{\xi}, t)\,|_{\Omega^e} = \sum_{a=1}^{4} \mathbf{B}_a(\boldsymbol{\xi})\,\boldsymbol{\theta}_a(t), \qquad \mathbf{B}_a = \begin{bmatrix} 0 & -N_{a,x_1} \\ N_{a,x_2} & 0 \\ N_{a,x_1} & -N_{a,x_2} \end{bmatrix}, \tag{33}$$

where notation $N_{a,x_i} = \frac{\partial N_a}{\partial \xi^j}\frac{\partial \xi^j}{\partial x_i}$ has been used. We further choose a bilinear distribution over the element for the shear strains

$$\begin{pmatrix} \gamma_1^h \\ \gamma_2^h \end{pmatrix} = \boldsymbol{\gamma}^h(\boldsymbol{\xi}, t)\,|_{\Omega^e} = \sum_{I=1}^{4} N_I(\boldsymbol{\xi})\,\boldsymbol{\gamma}_I(t), \qquad \boldsymbol{\gamma}_I = [\gamma_{1I}, \gamma_{2I}]^T, \tag{34}$$

where the nodal shear strains $\boldsymbol{\gamma}_I$ are obtained from (3), (30) and (31) as

$$\boldsymbol{\gamma}_I = \frac{1}{\mathbf{t}_{IJ}^T \mathbf{n}_{IK}} \begin{bmatrix} \frac{1}{l_{IK}}\mathbf{n}_{IJ} w_K + \frac{1}{l_{IJ}}\mathbf{n}_{IK} w_J - \left(\frac{1}{l_{IK}}\mathbf{n}_{IJ} + \frac{1}{l_{IJ}}\mathbf{n}_{IK}\right) w_I + \\ \frac{1}{2}\mathbf{n}_{IJ}\mathbf{n}_{IK}^T \boldsymbol{\theta}_K - \frac{1}{2}\mathbf{n}_{IK}\mathbf{n}_{IJ}^T \boldsymbol{\theta}_J + \frac{1}{2}\left(\mathbf{n}_{IJ}\mathbf{n}_{IK}^T - \mathbf{n}_{IK}\mathbf{n}_{IJ}^T\right)\boldsymbol{\theta}_I \end{bmatrix}, \qquad \begin{array}{c|cccc} I & 1 & 2 & 3 & 4 \\ \hline J & 4 & 1 & 2 & 3 \\ \hline K & 2 & 3 & 4 & 1 \end{array}. \tag{35}$$

The notation for strains and transverse displacement can be further simplified as

$$\begin{aligned} \boldsymbol{\kappa}^h(\boldsymbol{\xi}, t)\,|_{\Omega^e} &= \sum_{a=1}^{4} \widetilde{\mathbf{B}}_a(\boldsymbol{\xi})\,\mathbf{u}_a(t), \qquad \boldsymbol{\gamma}^h(\boldsymbol{\xi}, t)\,|_{\Omega^e} = \sum_{a=1}^{4} \mathbf{G}_a(\boldsymbol{\xi})\,\mathbf{u}_a(t), \\ w^h(\boldsymbol{\xi}, t)\,|_{\Omega^e} &= \sum_{a=1}^{4} \mathbf{M}_a(\boldsymbol{\xi})\,\mathbf{u}_a(t), \qquad \mathbf{u}_a = \begin{pmatrix} w_a \\ \boldsymbol{\theta}_a \end{pmatrix}, \end{aligned} \tag{36}$$

where $\widetilde{\mathbf{B}}_a$ follows from (33), $\mathbf{G}_a$ from (34) and (35) and $\mathbf{M}_a$ from (31). The virtual quantities $\widehat{\boldsymbol{\kappa}}^h(\boldsymbol{\xi})$, $\widehat{\boldsymbol{\gamma}}^h(\boldsymbol{\xi})$ and $\widehat{w}^h(\boldsymbol{\xi})$ are interpolated in the same manner as corresponding quantities in (36). One can also introduce more compact notation: $\boldsymbol{\varepsilon}^h = \left[\boldsymbol{\kappa}^{h,T}, \boldsymbol{\gamma}^{h,T}\right]^T = \widetilde{\widetilde{\mathbf{B}}}_a^T \mathbf{u}_a$, $\widehat{\boldsymbol{\varepsilon}}^h = \widetilde{\widetilde{\mathbf{B}}}_a^T \widehat{\mathbf{u}}_a$, $\widetilde{\widetilde{\mathbf{B}}}_a^T = \left[\widetilde{\mathbf{B}}_a^T, \mathbf{G}_a^T\right]$.

When the above interpolations are introduced in the weak form of equilibrium equations (4) one gets for an element (e) the following discretized equation (we assume that only load $p = p(\boldsymbol{\xi}, t)$ is active)

$$G^{(e)}\left(w^h(\boldsymbol{\xi},t), \boldsymbol{\theta}^h(\boldsymbol{\xi},t); \widehat{w}_a, \widehat{\boldsymbol{\theta}}_a\right) = \sum_{a=1}^{4} \widehat{\mathbf{u}}_a^{(e),T}\left(\widehat{w}_a, \widehat{\boldsymbol{\theta}}_a\right) \mathbf{r}_a^{(e)}\left(w^h(\boldsymbol{\xi},t), \boldsymbol{\theta}^h(\boldsymbol{\xi},t)\right), \tag{37}$$

where $\widehat{\mathbf{u}}_a = \left[\widehat{w}_a, \widehat{\boldsymbol{\theta}}_a^T\right]^T$,

$$\begin{aligned}\mathbf{r}_a^{(e)} = &\int_{\Omega^{(e)}} \widetilde{\widetilde{\mathbf{B}}}_a^T(\boldsymbol{\xi})\, \boldsymbol{\sigma}\left(\boldsymbol{\varepsilon}^h(\boldsymbol{\xi},t), \boldsymbol{\varepsilon}^p(\boldsymbol{\xi},t), \xi(\boldsymbol{\xi},t), \varkappa(\boldsymbol{\xi},t)\right) d\Omega^{(e)} - \\ &\int_{\Omega^{(e)}} \mathbf{M}_a^T(\boldsymbol{\xi})\, \mathbf{p}(\boldsymbol{\xi},t)\, d\Omega^{(e)}\end{aligned}, \tag{38}$$

$\mathbf{p} = \left[p^h, 0, 0\right]^T$, $\boldsymbol{\varepsilon}^h(\boldsymbol{\xi},t) = \boldsymbol{\varepsilon}^h\left(\boldsymbol{\theta}^h(\boldsymbol{\xi},t),\, \boldsymbol{\gamma}^h(\boldsymbol{\xi},t)\right)$, etc. Numerical integration of (38) (2×2 Gauss integration points are used for the present element) leads to

$$\mathbf{r}_a^{(e)} = \sum_{G=1}^{4} W_G \left(\begin{array}{l}\widetilde{\widetilde{\mathbf{B}}}_a^T(\boldsymbol{\xi}_G)\, \boldsymbol{\sigma}\left(\boldsymbol{\varepsilon}^h(\boldsymbol{\xi}_G,t), \boldsymbol{\varepsilon}^p(\boldsymbol{\xi}_G,t), \xi(\boldsymbol{\xi}_G,t), \varkappa(\boldsymbol{\xi}_G,t)\right) - \\ \mathbf{M}_a^T(\boldsymbol{\xi}_G)\, \mathbf{p}(\boldsymbol{\xi}_G,t)\end{array}\right) \det \mathbf{J}(\boldsymbol{\xi}_G), \tag{39}$$

where $\boldsymbol{\xi}_G$ are $\boldsymbol{\xi}$ coordinates evaluated at the Gauss point, W_G is Gauss point weight, and $\mathbf{J}$ is Jacobian matrix of the mapping $\boldsymbol{\xi} \mapsto \mathbf{x}^h$. It can be seen that the values of state variables need to be obtained only at the integration points for a particular value of pseudo-time. The component of the element consistent tangent stiffness matrix are

$$\mathbf{K}_{ab}^{(e)} = \frac{\partial \mathbf{r}_a^{(e)}}{\partial \mathbf{u}_b} = \sum_{G=1}^{4} W_G \widetilde{\widetilde{\mathbf{B}}}_a^T(\boldsymbol{\xi}_G) \underbrace{\frac{\partial \boldsymbol{\sigma}(\boldsymbol{\xi}_G,t)}{\partial \boldsymbol{\varepsilon}^h}}_{\dot{\boldsymbol{\sigma}}\dot{\boldsymbol{\varepsilon}}^{-1}(\boldsymbol{\xi}_G,t) = \mathbf{C}^{ep}(\boldsymbol{\xi}_G,t)} \underbrace{\frac{\partial \boldsymbol{\varepsilon}^h(\boldsymbol{\xi}_G,t)}{\partial \mathbf{u}_b}}_{\widetilde{\widetilde{\mathbf{B}}}_b(\boldsymbol{\xi}_G)} \det \mathbf{J}(\boldsymbol{\xi}_G). \tag{40}$$

The element consistent stiffness matrix and the element residual vector follow from (39) and (40) as

$$\mathbf{K}^{(e)} = \left[\mathbf{K}_{ab}^{(e)}\right], \qquad \mathbf{r}^{(e)} = \left[\mathbf{r}_a^{(e),T}\right]^T, \qquad a,b = 1,2,3,4\ . \tag{41}$$

The assembly procedure follows the usual approach explained in the finite element textbooks, e.g. [8]. The resulting nonlinear equations for nodal displacements/rotations of the chosen finite element mesh are solved by incremental/iterative Newton-Raphson solution procedure.

3.2 Computational issues for plasticity

As a result of space discretization, addressed in the previous section, the evolution equations (20) become ordinary differential equations in time that need to be integrated numerically at each integration point. Backward Euler integration scheme is used for that end. The solution is searched for at discrete pseudo-time points $0 < t_1 < \ldots t_n < t_{n+1} \ldots < T$. At a typical pseudo-time increment $\Delta t = t_{n+1} - t_n$ and typical integration point located at $\mathbf{x}^h(\boldsymbol{\xi}_G) \in \Omega^e$ the problem can be stated as: By knowing the values of the internal variables at the beginning of the pseudo-time increment, i.e. $\boldsymbol{\varepsilon}_n^p, \xi_n,\ \varkappa_n$, find values of the internal variables at the end of the pseudo-time increment, i.e. $\boldsymbol{\varepsilon}_{n+1}^p, \xi_{n+1},\ \varkappa_{n+1}$, which should satisfy the yield criterion. In the spirit of the operator split method [8] one assumes that the best iterative guess for strains at the end of the pseudo-time increment, $\boldsymbol{\varepsilon}_{n+1}^{(i)}$, is given data. Here (i) is iteration counter of the (global) Newton-Raphson solution procedure.

Prior to the integration of evolution equations the following test is performed: assume that the pseudo-time step from t_n to t_{n+1} remains elastic and evaluate the trial (test) values of strain-like and stress-like internal variables

$$\boldsymbol{\sigma}_{n+1}^{trial} = \mathbf{C}\left(\boldsymbol{\varepsilon}_{n+1}^{(i)} - \underbrace{\boldsymbol{\varepsilon}_{n+1}^{p,trial}}_{\boldsymbol{\varepsilon}_n^p}\right), \qquad q_{n+1}^{trial} = -\Xi'\left(\underbrace{\xi_{n+1}^{trial}}_{\xi_n}\right), \qquad \boldsymbol{\alpha}_{n+1}^{trial} = -\frac{2}{3}H_{kin}\underbrace{\varkappa_{n+1}^{trial}}_{\varkappa_n}. \tag{42}$$

If the yield function evaluated with those trial variables $\phi_{n+1}^{tr} = \phi\left(\boldsymbol{\sigma}_{n+1}^{trial}, q_{n+1}^{trial}, \boldsymbol{\alpha}_{n+1}^{trial}\right) \leq 0$, then, see (21), $\gamma_{n+1} = \dot{\gamma}\Delta t = 0$. The final values at the end of the pseudo-time increment (marked with the bar) equal the trial values, i.e. $\overline{\boldsymbol{\varepsilon}}_{n+1}^p = \boldsymbol{\varepsilon}_{n+1}^{p,trial}$, $\overline{\xi}_{n+1} = \xi_{n+1}^{trial}$ and $\overline{\varkappa}_{n+1} = \varkappa_{n+1}^{trial}$. The pseudo-time step is indeed elastic.

In the case that the yield function for those trial variables is violated, then $\gamma_{n+1} > 0$ and $\phi_{n+1} = 0$. Backward Euler integration of evolution equations is performed, i.e.

$$\boldsymbol{\varepsilon}_{n+1}^p = \boldsymbol{\varepsilon}_n^p + \gamma_{n+1}\boldsymbol{\nu}_{n+1}, \qquad \xi_{n+1} = \xi_n + \gamma_{n+1}\beta_{n+1}, \qquad \varkappa_{n+1} = \varkappa_n + \gamma_{n+1}\boldsymbol{\nu}_{n+1}. \tag{43}$$

Equations (43) and $\phi_{n+1} = 0$ can be written as the following set of nonlinear equations (with respect to $\boldsymbol{\varepsilon}_{n+1}^p = \varkappa_{n+1}$, ξ_{n+1} and γ_{n+1})

$$\begin{gathered}
\boldsymbol{\varepsilon}_{n+1}^p - \boldsymbol{\varepsilon}_n^p - \gamma_{n+1}2\mathbf{A}\left(\mathbf{C}\left(\boldsymbol{\varepsilon}_{n+1}^{(i)} - \boldsymbol{\varepsilon}_{n+1}^p\right) - \frac{2}{3}H_{kin}\boldsymbol{\varepsilon}_{n+1}^p\right) = \mathbf{0}, \\
\xi_{n+1} - \xi_n - \gamma_{n+1}\frac{2}{\sigma_y}\left(1 + \frac{\Xi'\left(\xi_{n+1}\right)}{\sigma_y}\right) = 0, \\
\left(\mathbf{C}\left(\boldsymbol{\varepsilon}_{n+1}^{(i)} - \boldsymbol{\varepsilon}_{n+1}^p\right) - \frac{2}{3}H_{kin}\boldsymbol{\varepsilon}_{n+1}^p\right)^T \mathbf{A}\left(\mathbf{C}\left(\boldsymbol{\varepsilon}_{n+1}^{(i)} - \boldsymbol{\varepsilon}_{n+1}^p\right) - \frac{2}{3}H_{kin}\boldsymbol{\varepsilon}_{n+1}^p\right) - \left(1 + \frac{\Xi'\left(\xi_{n+1}\right)}{\sigma_y}\right)^2 = 0,
\end{gathered} \tag{44}$$

which can be solved iteratively by Newton procedure to get the final values $\overline{\boldsymbol{\varepsilon}}_{n+1}^p$, $\overline{\xi}_{n+1}$ and $\overline{\gamma}_{n+1}$.

For more effective solution of (43) one can write relations (16) and (17) as, see (20)

$$\begin{aligned}
\boldsymbol{\sigma}_{n+1} &= \mathbf{C}\left(\boldsymbol{\varepsilon}_{n+1}^{(i)} - \boldsymbol{\varepsilon}_{n+1}^{p,trial}\right) = \boldsymbol{\sigma}_{n+1}^{trial} - \gamma_{n+1}\mathbf{C}\underbrace{2\mathbf{A}\left(\boldsymbol{\sigma}_{n+1} + \boldsymbol{\alpha}_{n+1}\right)}_{\boldsymbol{\nu}_{n+1}}, \\
q_{n+1} &= -\Xi'(\xi_{n+1}), \qquad \xi_{n+1} = \xi_n + \gamma_{n+1}\underbrace{\frac{2}{\sigma_y}\sqrt{\left(\boldsymbol{\sigma}_{n+1} + \boldsymbol{\alpha}_{n+1}\right)^T \mathbf{A}\left(\boldsymbol{\sigma}_{n+1} + \boldsymbol{\alpha}_{n+1}\right)}}_{\beta_{n+1}}, \\
\boldsymbol{\alpha}_{n+1} &= -\frac{2}{3}H_{kin}\varkappa_{n+1} = \boldsymbol{\alpha}_{n+1}^{trial} - \gamma_{n+1}\frac{2}{3}H_{kin}\underbrace{2\mathbf{A}\left(\boldsymbol{\sigma}_{n+1} + \boldsymbol{\alpha}_{n+1}\right)}_{\boldsymbol{\nu}_{n+1}},
\end{aligned} \tag{45}$$

which leads to

$$\left(\boldsymbol{\sigma}_{n+1} + \boldsymbol{\alpha}_{n+1}\right) = \underbrace{\left[\mathbf{I}_5 + \gamma_{n+1}\left(2\mathbf{CA} + \frac{4}{3}H_{kin}\mathbf{A}\right)\right]^{-1}}_{\mathbf{W}_{n+1}(\gamma_{n+1})}\left(\boldsymbol{\sigma}_{n+1}^{trial} + \boldsymbol{\alpha}_{n+1}^{trial}\right), \qquad q_{n+1} = q_{n+1}\left(\gamma_{n+1}\right). \tag{46}$$

A closed form expression for the inverse of the matrix in (46) can be obtained by using spectral decomposition of $\mathbf{C}$ and $\mathbf{A}$; the procedure is very similar to the one at the plane stress situation, see [10], [21], [22], [13], and will not be repeated here. Explicit inversion of $\mathbf{W}_{n+1}$ enables expressing $(\boldsymbol{\sigma}_{n+1} + \boldsymbol{\alpha}_{n+1})$ in terms of single unknown γ_{n+1}. Since $\gamma_{n+1}\phi_{n+1} = 0$, see (21), one gets a single nonlinear equation in terms of γ_{n+1}

$$\phi_{n+1}(\left(\boldsymbol{\sigma}_{n+1} + \boldsymbol{\alpha}_{n+1}\right)\left(\gamma_{n+1}\right), q_{n+1}\left(\gamma_{n+1}\right)) = \phi_{n+1}\left(\gamma_{n+1}\right) = 0. \tag{47}$$

The solution of (47) is obtained by Newton iterative procedure

$$-\phi_{n+1}^{\prime(k)}\Delta\gamma_{n+1}^{(k)} = \phi_{n+1}^{(k)}, \qquad \gamma_{n+1}^{(k+1)} = \gamma_{n+1}^{(k)} + \Delta\gamma_{n+1}^{(k)}, \tag{48}$$

where (k) is iteration counter and $\phi'_{n+1} = \frac{d\phi_{n+1}}{d\gamma_{n+1}}$. The final (converged) solution of (47) is marked by the bar, i.e. $\overline{\gamma}_{n+1}$. The final values at the end of the pseudo-time increment are also marked with the bar; for example $\overline{\boldsymbol{\varepsilon}}^p_{n+1} = \overline{\varkappa}_{n+1}$ and $\overline{\xi}_{n+1}$ can be computed from (46) and (43) by using $\overline{\gamma}_{n+1}$.

Remark 4: Numerical experiments show that for the present case the function $\phi^{(k)}_{n+1}$ may be very steep; i.e. very large differences in function value can be obtained for small differences in function argument. However, no difficulties in computation of the numerical examples presented below were observed if the convergence criterion was based on the value of $\phi^{(k)}_{n+1}$; we used $\phi^{(k)}_{n+1} < 10^{-12}$ for convergence criterion. □

Remark 5: When yield function with $\alpha(t)$, see (13), was used in the numerical computations, the equivalent plastic curvature (14) of the pseudo-time increment $[t_n, t_{n+1}]$ was evaluated with $\check{\kappa}^p_n$ and α_n. The $\check{\kappa}^p_{n+1}$ for the next time step was calculated with numerical integration of equation (14)

$$\check{\kappa}^p_{n+1} = \check{\kappa}^p_n + \frac{Eh}{\sqrt{3}\sigma_y}\left[(\Delta\overline{\kappa}^p_{11,n+1})^2 + (\Delta\overline{\kappa}^p_{22,n+1})^2 + \Delta\overline{\kappa}^p_{11},_{n+1}\Delta\overline{\kappa}^p_{22},_{n+1} + (\Delta\overline{\kappa}^p_{12,n+1})^2/4\right]^{\frac{1}{2}}, \tag{49}$$

where $\Delta\overline{\kappa}^p_{11,n+1}, \Delta\overline{\kappa}^p_{22,n+1}, \Delta\overline{\kappa}^p_{12,n+1}$ are the first three components of $\Delta\overline{\boldsymbol{\varepsilon}}^p_{n+1} = \overline{\gamma}_{n+1}\overline{\boldsymbol{\nu}}_{n+1}$.□

To complete the elastoplastic implementation issues the consistent tangent matrix $d\boldsymbol{\sigma}_{n+1}/d\boldsymbol{\varepsilon}^{(i)}_{n+1}$ has to be derived for $\gamma_{n+1} > 0$. It is obtained by differentiation of $\boldsymbol{\varepsilon}^{(i)}_{n+1} = \mathbf{C}^{-1}\boldsymbol{\sigma}_{n+1} + \boldsymbol{\varepsilon}^p_{n+1}$, eqs. (43), and consistency condition $d\phi_{n+1} = 0$. After some manipulation one can have

$$d\boldsymbol{\sigma}_{n+1} = \left[\mathbf{C}^{-1} + 2\overline{\gamma}_{n+1}\mathbf{H}_{n+1}\mathbf{A} + \frac{3}{2\overline{f}_{n+1}H_{kin} + 6\overline{c}_{n+1}}\mathbf{H}_{n+1}\overline{\boldsymbol{\nu}}_{n+1}\overline{\boldsymbol{\nu}}^T_{n+1}\mathbf{H}_{n+1}\right]^{-1} d\boldsymbol{\varepsilon}^{(i)}_{n+1}, \tag{50}$$

where

$$\mathbf{H}^{-1}_{n+1} = \mathbf{I}_5 + \frac{4}{3}\gamma_{n+1}H_{kin}\mathbf{A}, \qquad \overline{f}_{n+1} = \overline{\boldsymbol{\nu}}^T_{n+1}\mathbf{H}_{n+1}\overline{\boldsymbol{\nu}}_{n+1}, \qquad \overline{c}_{n+1} = \frac{2\left(1 + \frac{\Xi'(\overline{\xi}_{n+1})}{\sigma_y}\right)^2 \Xi''\left(\overline{\xi}_{n+1}\right)}{\sigma_y^2 - 2\overline{\gamma}_{n+1}\Xi''\left(\overline{\xi}_{n+1}\right)}. \tag{51}$$

The inverse of $\mathbf{H}^{-1}_{n+1}$ can be easily obtained in closed form. The matrix in (50) can be obtained in closed form as well by using Sherman-Morrison formula, see e.g. [18].

3.3 Computational issues for viscoplasticity

The above discussed viscoplastic plate model allows one to define a unified framework for both stress resultant elastoplasticity and stress resultant viscoplasticity for plates. Namely, the integration procedure for the plate viscoplasticity is essentially the same as for the plate plasticity, except that for $\phi^{tr}_{n+1} > 0$ one looks for $\gamma_{n+1} = \frac{\Delta t}{\eta}\left\langle\phi_{n+1}\right\rangle > 0$. Its final value is obtained by iterative solution of nonlinear equation

$$-\frac{\eta}{\Delta t}\gamma_{n+1} + \phi_{n+1}\left(\gamma_{n+1}\right) = 0 \rightarrow \overline{\gamma}_{n+1}. \tag{52}$$

The consistent tangent matrix is obtained in the same manner as for plasticity except that one has to replace in its derivation the consistency condition $d\phi_{n+1} = 0$ by $d\phi_{n+1} - \frac{\eta}{\Delta t}d\gamma_{n+1} = 0$. The form of the consistent tangent matrix is the same as (50) except that $\overline{c}_{n+1}$ is replaced by $\overline{c}_{n+1} + \frac{\eta}{\Delta t}$.

4 Numerical examples

The finite element code for inelastic plate analysis was generated by using symbolic code manipulation program AceGen developed by Korelc [16] and implemented into the finite element analysis program AceFEM, see Korelc [15]. We note that the plate element used in this work is locking-free as shown in [9] (see results for PQ2 element).

4.1 Limit load analysis of a rectangular plate

A rectangular plate of elastic-perfectly plastic material under uniformly distributed load is analyzed for two sets of boundary conditions: simple supported (of hard type) and clamped (of hard type). The plate characteristics are: thickness $h = 0.5$ cm, length $l = 150$ cm, width $b = 100$ cm. Material parameters are: Young's modulus $E = 21000$ kN/cm^2, Poisson's ration $\nu = 0.3$ and yield stress $\sigma_y = 40$ kN/cm^2. Numerical analysis was performed with a coarse mesh of 8×8 and with a fine mesh of 60×40 elements, see Figure 6. We compare our results with those obtained by ABAQUS' [7] quadrilateral shell element (S4R element) with through-the-thickness stress integration (with 5 integration points) and von Mises yield criterion. Load-displacement curves are presented in Figures 2 and 3. There is a difference in results of both analyses since the stress resultant formulation does not account for gradual through-the-thickness plastification. However, equal limit load is obtained in both cases. It is interesting to see that the mesh density plays more important role in the accuracy of the limit load computation than the chosen way of definition of elastoplastic constitutive model. Namely, the difference between the coarse and fine mesh in predicting the limit load is around 20 % for the clamped plate, see Figure 3. By replacing m_0 with αm_0, we can estimate gradual spreading of plastic zones through the thickness. In Figures 4 and 5 we present load-displacement curves obtained by using α parameter and coarse mesh. We used a constant value of α across one time increment and therefore the yield criterion is no longer smooth in pseudo-time. To reduce the influence of this effect we used small time increments. In case of simply supported plate (Figure 4) the first yield is well predicted, yet the curve in subsequent states is below the ABAQUS' curve. Results for clamped plate are much better since one can hardly distinguish between ABAQUS and stress resultant formulation when using time increment $\Delta t = 0.0025$.

4.2 Limit load analysis of a circular plate

We analyze a uniformly loaded circular plate of the same elastic-perfectly plastic material as in previous example. Again we consider simply supported and clamped plates. The radius of the plate is $r = 50$ cm and the thickness is $h = 0.5$ cm. The meshes are shown in Figure 6. In Figures 7 and 8 we plot load-displacement curves. Again we see that the coarse mesh overestimates limit load in the case of clamped plate. Overall correspondence of two formulations is reasonable. In Tables 1 and 5 we compare our results for limit load with analytical solutions found in textbooks on plasticity [17], [19] and Eurocode [6]. Our result for simply supported plate is in complete agreement with solution based on von Mises yield criterion, see Table 1. In the case of clamped plate our result is slightly greater than the von Mises yield based solution, see Table 5.

4.3 Elastoplastic analysis of a skew plate

We consider a skew plate of elastic-plastic material with hardening under uniformly distributed load. The plate thickness is $h = 0.5$ cm, the longer side is $a = 150$ cm, the shorter one is $b = 135$ cm and the in-between angle is $\phi = 45°$. All material properties are the same as in the above examples, except for isotropic hardening modulus, which is now $H_{iso} = 0.1E = 2100 kN/cm^2$. The plate is supported along the shorter edges with five equally spaced point supports restraining displacements and allowing both rotations. Mesh is shown in Figure 9. Load versus centre displacement diagrams are presented in Figure 10. Both curves have similar shapes, yet the curve obtained with the coarse mesh is again above the curve obtained with the fine mesh. We see that the yielding of the plate significantly reduces its stiffness, yet the limit load is never reached because of the isotropic hardening. When using stress resultant plasticity model one can easily track the spreading of plastic zones. In Figure 11 we see that the yielding starts in the corners of the shorter diagonal, then it reaches the centre of the plate and spreads in the direction of longer diagonal corners.

4.4 Cyclic analysis of a circular plate

We consider a clamped circular plate under cyclic loading conditions. The plate is loaded with uniformly distributed load with the amplitude that corresponds to twice of the load at the first yield $p_{max} = 2\left(1.5(\frac{h}{r})^2\sigma_y\right)$. We examine three hardening cases: (i) isotropic hardening ($H_{iso} = 2100$ kN/cm^2, $H_{kin} = 0$ kN/cm^2), (ii) kinematic hardening ($H_{iso} = 0$ kN/cm^2, $H_{kin} = 2100$ kN/cm^2), (iii) combined isotropic and kinematic hardening ($H_{iso} = H_{kin} = 1050$ kN/cm^2). The remaining material and geometry parameters are the same as those adopted for the limit load analysis, except for plate thickness which is now $h = 4$ cm. In Figure 12 the load-displacement curves for the first two cycles are presented for stress resultant formulation and stress resultant formulation with parameter α. Isotropic hardening enlarges the yield surface which can be seen in Figure 12 where the curve appears as a closed loop after the first cycle. After the initial cycle the plate can sustain greater stress resultants and still remain elastic. The shift of the initial curve to the right represents the plastic deformation. Purely kinematic hardening curve is wider and is virtually unchanged from one cycle to another. In this case the size of the yield surface is unchanged whereas the effect of kinematic hardening changes the position of it. We can look at the combined hardening curve as a combination of the purely isotropic and purely kinematic hardening curves. Isotropic hardening effects prevail and after the first cycle the plate remains in elastic state.

4.5 Elastoviscoplastic analysis of a circular plate

In this example we consider a clamped circular plate of elastoviscoplastic material. All material and geometry parameters are the same as in the case of limit load analysis. Three different values of viscosity parameter are chosen, $\eta = 0$, $\eta = 1$, $\eta = 10$, for two sets of loading conditions. In the first set we gradually apply a point load in the centre of the plate until it reaches its final value $F = 22\ kN$ at time $t = 1$. The second set is displacement driven with a prescribed final value of midpoint deflection $w = 11\ cm$. Loading curves for both loading sets are presented in Figure 13. We show the time-deflection curve of the plate under first loading condition in Figure 14. The viscosity coefficient η has a significant effect on a nature of inelastic response. The value $\eta = 0$ corresponds to plasticity whereas for values $\eta > 0$ the inelastic deformations are time dependent. One can note that the strain in elastoviscoplastic material held at constant stress will gradually reach the level of strain in a time independent material. Time response of the plate for the strain driven loading is presented in Figure 15. We see a hardening like response in viscoplastic materials ($\eta > 0$) but resistance is slowly dropping to the value corresponding to the time independent material.

5 Concluding remarks

The stress resultant plasticity for plates has been revisited and reformulated. It has been systematically derived from the principle of maximum plastic dissipation and presented in a form suitable for effective computational implementation. Stress resultant overstress viscoplasticity has been introduced in such a manner that both inelastic formulations can be treated within one computational framework. We note that one could also use the same framework for power-law strain and strain-rate hardening viscoplasticity. Numerical results of the presented formulation have been compared with the stress formulation (ABAQUS) as well as with the stress resultant formulation with α parameter. It has been shown that, regarding the accuracy of the limit load computation, the mesh density plays more important role than the type of elastoplastic formulation. An extension of this work to geometrically nonlinear shells will be adressed in a separate work.

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Figure 1: Notation of the used finite element

Figure 2: Load - displacement diagram for simply supported rectangular plate

Limit load	Yield criterion	Reference
$q = 1.629\frac{h^2}{r^2}\sigma_y$	present	present (fine mesh)
$q = 1.625\frac{h^2}{r^2}\sigma_y$	-	[6]
$q = 1.500\frac{h^2}{r^2}\sigma_y$	Tresca (analytical solution)	[17], [19]
$q = 1.629\frac{h^2}{r^2}\sigma_y$	Von Mises (analytical solution)	[19]
$q = 2.000\frac{h^2}{r^2}\sigma_y$	Von Mises (analytical upper bound)	[19]
$q = 1.500\frac{h^2}{r^2}\sigma_y$	Von Mises (analytical lower bound)	[19]

Table 1: Limit load solutions for circular simply supported plate

Limit load	Yield criterion	Reference
$q = 3.240\frac{h^2}{r^2}\sigma_y$	present	present (fine mesh)
$q = 3.125\frac{h^2}{r^2}\sigma_y$	-	[6]
$q = 2.815\frac{h^2}{r^2}\sigma_y$	Tresca (analytical solution)	[17], [19]
$q = 3.138\frac{h^2}{r^2}\sigma_y$	Von Mises (analytical solution)	[19]

Figure 3: Load - displacement diagram for clamped rectangular plate

Figure 4: Load - displacement diagram for simply supported rectangular plate; α parameter case

Figure 5: Load - displacement diagram for clamped rectangular plate; α parameter case

Figure 6: Meshes used for: (a) rectangular plate - fine, (b) rectangular plate - coarse, (c) circular plate - fine and (d) circular plate - coarse

Figure 7: Simply supported circular plate - limit load analysis

Figure 8: Clamped circular plate - limit load analysis

Figure 9: Skew plate - (a) fine mesh, (b) coarse mesh

Figure 10: Skew plate - elastoplastic analysis

Figure 11: Spreading of plastic zones

Figure 12: Clamped circular plate - cyclic load

Figure 13: Loading curve for viscoplastic analyses

Figure 14: Time response for force-prescribed viscoplastic circular plate

Figure 15: Time response for displacement-prescribed viscoplastic circular plate