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IZJAVE

Skladno s 27. členom Pravilnika o diplomskem delu UL Fakultete za gradbeništvo in geodezijo, ki ga je sprejel Senat Fakultete za gradbeništvo in geodezijo Univerze v Ljubljani na svoji 10. seji dne 21. 4. 2010.

Podpisani Bojan Šavrič, izjavljam, da sem avtor diplomske naloge z naslovom: »*Določitev polinomske enačbe za Naravno Zemljino kartografsko projekcijo*«. Noben del te diplomske naloge ni bil uporabljen za pridobitev strokovnega naziva ali druge strokovne kvalifikacije na tej ali na drugi univerzi ali izobraževalni inštituciji.

Izjavljam, da prenašam vse materialne avtorske pravice v zvezi z diplomsko nalogo na UL Fakulteto za gradbeništvo in geodezijo. UL FGG ima ob pridobitvi izrecnega pisnega soglasja študenta in mentorja pravico do javne objave diplomske naloge.

Izjavljam, da dovoljujem objavo elektronske različice na spletnih straneh UL FGG in ETH Zürich, Institute of Cartography and Geoinformation IKG.

Izjavljam, da je elektronska različica v vsem enaka tiskani različici.

I, Bojan Šavrič, hereby declare that I am the author of the graduation thesis entitled: »*Derivation of a Polynomial Equation for the Natural Earth Projection*«.

I confirm that my work may be uploaded on the websites of UL FGG and ETH Zürich, Institute of Cartography and Geoinformation IKG.

Ljubljana, 31 August 2011

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Abstract

The Natural Earth projection is a new projection for representing the entire Earth on small-scale maps. It was designed in Flex Projector, a specialized software application that offers a graphical approach for the creation of new projections. The original Natural Earth projection defines the length and spacing of parallels in a tabular form for every five degrees of increased latitude. It is a true pseudo-cylindrical projection, and is neither conformal nor equal-area. In the original definition, piece-wise cubic spline interpolation is used to project intermediate values that do not align with the five-degree grid.

This graduation thesis introduces alternative polynomial equations that are considerably simpler to compute. The polynomial expression also improves the smoothness of the rounded corners where the meridians meet the horizontal pole lines, a distinguished mark of the Natural Earth projection which suggests to readers that the Earth is spherical in shape. An inverse projection is presented. The formulas are simple to implement in cartographic software and libraries. Distortion values of this new graticule are not significantly different from the original piece-wise projection.

The development of the polynomial equations was inspired by a similar study of the Robinson projection. The polynomial coefficients were determined with least square adjustment of indirect observations with additional constraints to preserve the height and width of the graticule. The inverse procedure uses the Newton-Raphson method and converges in a few iterations.

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Ključne besede: Naravna Zemljina projekcija, Flex Projector, Robinsonova projekcija, metoda najmanjših kvadratov, MNK, polinomska enačba, tangentna metoda, inverzna enačba

Izvleček

Naravna Zemljina projekcija (ang. The Natural Earth projection) je nova kartografska projekcija za karte celotnega sveta v majhnih merilih. Zasnovana je bila s pomočjo programa Flex Projector, iterativnega orodja za grafično oblikovanje in primerjavo kartografskih projekcij. Določena je na podlagi dolžine in razporeditve paralel za vsakih pet stopinj geografske širine. Vrednosti so podane tabelarično. Vmesne točke te 5-stopinjske mreže so izvorno določene z interpolacijsko metodo kubičnih zlepkov. Projekcija je psevdocilindrična in ni niti konformna niti ekvivalentna.

Diplomska naloga predstavlja novo polinomsko enačbo za Naravno Zemljino projekcijo, ki je enostavnejša za izračun. Obenem enačba izboljšuje tudi gladkost zakrivljenih robov tam, kjer je stičišče robnega meridiana in pola. Zakrivljeni robovi so posebnost Naravne Zemljine projekcije in tisti, ki nakazujejo, da je Zemlja okrogla. Polinomske enačbe je mogoče invertirati in so enostavne za uporabo v kartografskih programih in drugih elektronskih bazah ter aplikacijah. Distorzijski parametri izboljšane mreže se niso bistveno spremenili in ostajajo skoraj enaki.

Ideja o uporabi polinomskih enačb izhaja iz podobne rešitve za Robinsonovo projekcijo. Vrednosti posameznih polinomskih koeficientov so bile določene z metodo najmanjših kvadratov. Da smo ohranili velikost in širino osnovne projekcije, je bila uporabljena posredna metoda izravnave s funkcijsko odvisnimi spremenljivkami. Inverzna transformacija uporablja tangentno metodo reševanja enačb, ki kvadratično konvergira h končni rešitvi.

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First of all, I would like to thank my supervisor Assist. Prof. Dr. Dušan Petrovič and co-advisors Dr. Bernhard Jenny and Prof. Dr. Lorenz Hurni for their expert guidance, encouragement and support while I was writing my graduation thesis, and their assistance and organization of my exchange study. I am also thankful to Tom Patterson for his collaboration during my research and graphical evaluation of the results, and to the employees of Institute of Cartography and Geoinformation, the Swiss Federal Institute of Technology (ETH) Zürich, who made me feel welcome and offered their help during my stay there. My words of gratitude go to my parents and brother – thank you for supporting me during my studies and for accepting all my decisions, even though they were sometimes difficult for you. I am grateful to Mirjana and others for unconditionally believing in me in challenging times when I doubted myself. Last but not least, I would like to thank my study colleagues, who always motivated me. I learned a lot from you, which helped me enter the branch of geodesy.

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This graduation thesis was written under the supervision and support of my supervisor Assist. Prof. Dr. Dušan Petrovič, University of Ljubljana and co-advisers Dr. Bernhard Jenny and Prof. Dr. Lorenz Hurni both ETH Zürich. All work on my thesis, including research and writing, was done at Swiss Federal Institute of Technology (ETH) Zürich, Institute of Cartography and Geoinformation IKG, where I obtained all support and guidance I needed. I was studying there as an exchange student from February till June 2011.

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O NASTANKU DIPLOMSKE NALOGE

Diplomsko nalogo sem izdelal pod mentorstvom doc. dr. Dušana Petroviča z Univerze v Ljubljani ter somentorstvom dr. Bernharda Jennyja in prof. dr. Lorenza Hurnija z ETH Zürich. Celotno delo za svojo diplomsko nalogo, raziskavo in pisanje sem opravil na Švicarski tehnični univerzi (ETH) v Zürichu, na Inštitutu za kartografijo in geoinformatiko, kjer sem dobil vso podporo in nasvete, ki sem jih potreboval. Svoj študij na omenjeni instituciji sem opravljal kot študent na izmenjavi od februarja do junija 2011.

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1 INTRODUCTION

In this thesis the derivation of a polynomial equation for the Natural Earth projection is presented. This introduction contains the problem overview and describes the main goal, the determination of the analytical expression for this projection. The second goal, smoothing the corners at the pole lines, that improves the studied projection, is presented as well. Finally, the structure of this graduation thesis is listed.

1.1 Problem overview: The Natural Earth projection

The creation of new map projections by cartographers was difficult mainly because of the general lack of mathematical expertise and the inability to preview the developed graticule. Until Arthur H. Robinson presented a graphic approach for developing pseudo-cylindrical projections with tabular parameters (Robinson, 1974), all projections were developed by analytical equations.

Robinson's new approach simplified the development of projections, but it does not provide a normal analytical expression relating longitude / latitude to Cartesian coordinates. Instead of an analytical equation, his projection was defined by a set of control points. Robinson did not reveal which interpolation method was used through these points to project intermediate points. For this reason, there are many different interpolation algorithms for the Robinson projection (Ipbüker, 2005).

An interactive tool for the graphical design and evaluation of map projections that uses Robinson's approach is the software applications Flex Projector. According to Jenny et al. (2008), the principal goal of Flex Projector is to provide tools to expert mapmakers for designing new map projections. Until now, three projections designed by this application have been published: the A4, the Cropped Ginzburg VIII and the Natural Earth projection (Jenny et al., 2008, 2010).

Tom Patterson developed the Natural Earth projection in 2007 by using Flex Projector (Jenny et al., 2008). Since this free, open-source, and cross-platform software application uses Robinson's approach, all designed projections – including the Natural Earth projection – do not have normal analytical expression. Projections are defined by two tabular parameters: the length of parallels and the distance of parallels from the equator, in sets of 19 control points for each five degrees of positive latitude. The original Natural Earth projection uses a cubic spline interpolation for each piece of five degrees. Interpolation is quick to evaluate but it requires a large number of parameters, and its implementation into geospatial software requires considerable effort. This barrier prevents its widespread use and because of that, the only software application where the Natural Earth projection can be computed is Flex Projector, where the projection was designed.

1.2 Main goal: analytical expression

In order to spread the usage of the Natural Earth projection to other software and projection libraries, the main goal of this graduation thesis is the derivation of an analytical expression of the projection. The expression should be a direct transformation of spherical coordinates to Cartesian coordinates. To simplify the transformation, equations should have a minimum number of parameters, and an inverse function should also be available. The implementation of the forward and inverse projection must be simple.

1.3 Second goal: smoothing the corners

The Natural Earth projection is distinguished by its rounded corners, where meridians meet the pole line. However, corners of the original Natural Earth projection are not completely smooth. At high magnification small edges can be seen where the pole line ends (Figure 1). In Flex Projector, the five degree spacing between latitude lines did not provide enough refinement to smooth these corners completely. The designer of the projection, Tom Patterson, expressed his desire to smooth these corners.

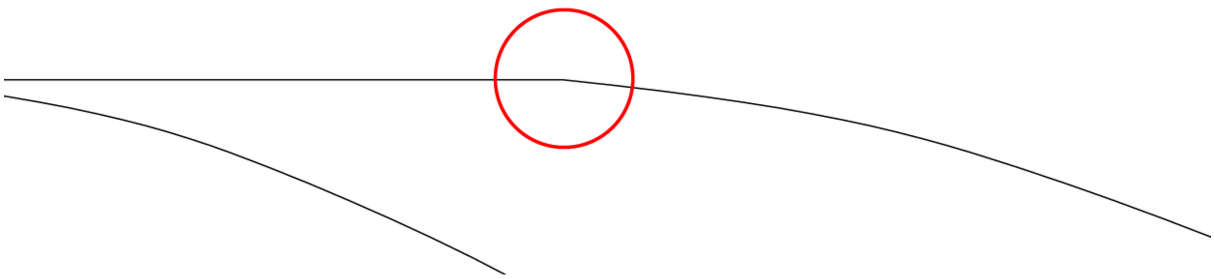


Figure 1: High magnification of the rounded corners shows a small edge (in red circle) at the end of the pole line.

The second goal of this thesis is to improve the smoothness of the rounded corners where the border meridians meet the pole lines. The smoothness of these corners should make sure that the edges are not visible any more.

1.4 Organization of the graduation thesis

The thesis is structured as follows. Chapter 2 introduces the theoretical background of the Natural Earth projection and different approaches to solve the same challenge for the Robinson projection. Chapter 3 describes the approximation method using least squares adjustment, and the Newton-Raphson iteration method for the inverse projection. The development of the polynomial projection is presented in Chapter 4. It describes the derivation of the polynomial equations and the improvement of the smoothness of the rounded corners. Chapter 5 presents the results of the thesis consisting in forward and inverse projections for transforming spherical coordinates to Cartesian X/Y coordinates and vice versa. Chapter 6 evaluates the improved and the original Natural Earth projections.

Three appendices are added to the thesis. Appendices A and B contain the Matlab code for calculating the polynomial coefficients for the forward projection, and the function for the inverse projection written for the purpose of this thesis. Appendix C contains Java code for implementing the Natural Earth projection in Java applications. This code is included in the Java Map Projection Library (Jenny, 2011).

This graduation thesis is an expansion of the paper "A polynomial equation for the Natural Earth projection" by Bojan Šavrič, Bernhard Jenny, Tom Patterson, Dušan Petrovič and Lorenz Hurni, accepted for the publication in the journal Cartography and Geographic Information Science (Šavrič et al., accepted for publication).

2 DESIGN AND ORIGIN OF THE NATURAL EARTH PROJECTION

This chapter contains the theoretical background of the studied projection and related work. Firstly, the design procedure of the projection in Flex Projector is described, and the definition of the original Natural Earth projection is presented. The second section describes the characteristics of the Natural Earth projection. Since the Robinson projection was designed with the same graphical approach, this projection does not have an analytical expression. Therefore a short overview of the methods for finding the analytical expressions for the Robinson projection is given at the end of this chapter.

2.1 Design of the Natural Earth projection

The Natural Earth projection was developed by Tom Patterson in 2007 out of his dissatisfaction with existing projections for displaying physical data on small-scale world maps (Jenny et al., 2008). Flex Projector, a freeware application for the interactive design and evaluation of map projections, was used as the means for creating the Natural Earth projection. The graphical user interface in Flex Projector allows cartographers to adjust the length, shape, and spacing of parallels and meridians of new projections in a graphical design process (Jenny and Patterson, 2007).

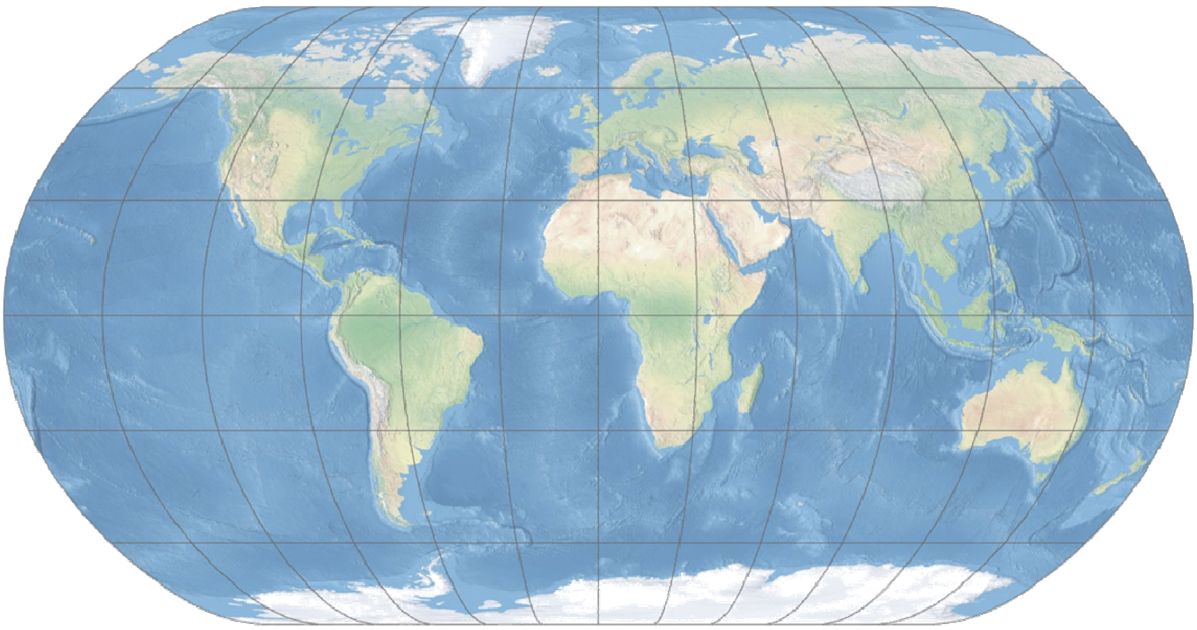


Figure 2: The Natural Earth projection applied to the Natural Earth dataset (Jenny, et al., 2008, page 68).

The Natural Earth projection is an amalgam of the Kavraiskiy VII and Robinson projections, with additional enhancements (Figure 2). These two projections most closely fulfilled the requirement for representing small-scale physical data on world maps, however, each had at least one undesirable characteristic (Jenny et al., 2008). The Kavraiskiy VII projection exaggerates the size of high latitude areas, resulting in oversized representation of polar regions. The Robinson projection, on the other hand, has a height-to-width ratio close to 0.5, resulting in a slightly too wide graticule with outward bulging sides and too much shape distortion near map edges.

Creating the Natural Earth projection required three major adjustments: Firstly, starting from the Robinson projection, its vertical extension slightly increased the height-to-width ratio from 0.5072 to 0.52 to give it more height. Secondly, using the Kavraiskiy VII as a template, the parallels were slightly increased in length. And thirdly, the length of the pole lines was decreased by a small amount to give the corners at pole lines a rounded appearance. Designing the Natural Earth projection in this way required trial-and-error experimentation and visual assessment of the appearance of continents in an iterative process (Jenny et al., 2008).

Table 1: Parameters of the Natural Earth projection (Jenny et al., 2008, page 25).

Latitude	Length of Parallels	Distance of Parallels from Equator
0	1	0
5	0.9988	0.062
10	0.9953	0.124
15	0.9894	0.186
20	0.9811	0.248
25	0.9703	0.31
30	0.957	0.372
35	0.9409	0.434
40	0.9222	0.4958
45	0.9006	0.5571
50	0.8763	0.6176
55	0.8492	0.6769
60	0.8196	0.7346
65	0.7874	0.7903
70	0.7525	0.8435
75	0.716	0.8936
80	0.6754	0.9394
85	0.627	0.9761
90	0.563	1
Height / width		0.52
Scale		0.8707

The shape of the graticule of any projection designed with Flex Projector is defined by tabular sets of parameters. For the Natural Earth projection, two parameter sets are used for specifying (1) the relative length of the parallels, and (2) the relative distance of parallels from the equator (Table 1). Equation 1 defines the original Natural Earth projection, transforming spherical coordinates to Cartesian X/Y coordinates (Jenny et al., 2008, 2010):

$$X = R \cdot s \cdot l_\varphi \cdot \lambda, \qquad l_\varphi \in [0,1],$$
$$Y = R \cdot s \cdot d_\varphi \cdot k \cdot \pi, \qquad d_\varphi \in [-1,1],$$ (Equation 1)

where:

X and Y are projected coordinates,

R is the radius of the generating globe,

$s = 0.8707$ is an internal scale factor,

l_φ is the relative length of the parallel at latitude φ, with $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $l_\varphi = 1$ for the equator, and with the slope of l_φ is 63.883° at the poles,

d_φ is the vertical distance of the parallel at latitude φ from the equator, with $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and with $d_\varphi = \pm 1$ for the pole lines, and $d_\varphi = 0$ for the equator,

λ is the longitude with $\lambda \in [-\pi, \pi]$ and

$k = 0.52$ is the height-to-width ratio of the projection.

Arthur H. Robinson proposed the structure of Equation 1 and the associated graphical approach to the design of small-scale map projections when he developed his eponymous projection (Robinson, 1974). In making the Natural Earth projection, Jenny et al. (2010) provide numerical values for the tabular parameters that define l_φ and d_φ in Equation 1 for each five-degree value (Table 1). For intermediate spherical coordinates that do not align with the five-degree grid, values for l_φ and d_φ need to be interpolated. The Flex Projector application uses a piece-wise cubic spline interpolation, with each piece of the spline curve covering five degrees. While this type of interpolation can be evaluated quickly, it is relatively intricate to program and requires a large number of parameters – factors that are likely to impede the widespread implementation of the Natural Earth projection in geospatial software. Seeking greater efficiency, the connection between parameters in Table 1 and spherical coordinates must be found in a compact analytical expression that approximates Equation 1.

2.2 The characteristics of the Natural Earth projection

The result of the procedure described in the previous section, the Natural Earth projection (Figure 2), is a true pseudo-cylindrical projection, i.e., a projection with regularly distributed meridians and straight parallels (Snyder, 1993, p. 189). The projection of the equator line defines the X axis, and the central meridian defines the Y axis (Figure P. 2, on page 38). The graticule is symmetric about the central meridian and the equator. The equator is 0.8707 times longer than the circumference of a sphere of equal area. The central meridian is a straight line 0.52 times longer than the equator. Other meridians are equally spaced elliptical arcs and do not intersect the parallels at straight angles. They are concave toward the central meridian. The corners where pole lines and bounding meridians meet have a rounded appearance, which is a distinguished characteristic among pseudo-cylindrical projections. Other meridians have a less rounded appearance, the closer they are to the central meridian the more they resemble a straight line.

As a compromise projection, the Natural Earth projection is neither a conformal nor equal area, but its distortion characteristics are comparable to, other well known projections. Its distortion values fall somewhere between those of the Kavraiskiy VII and Robinson projection that were used in the design procedure. Similar to all compromise projections, the Natural Earth projection also exaggerates the size of high latitude areas (Jenny et al., 2008). Figures 3, 4 and 5 illustrate these characteristics.

In Figure 3 the Tissot's indicatrices are presented for each 30 degrees. With larger distance from the equator, the area of indicatrices increases. The largest indicatrices on parallels of 60 degrees indicate that the projection exaggerates the size of high latitude areas. Since the size of indicatrices is enormous at pole lines, they are not presented in this figure. The axes of the indicatrices do not coincide with the directions of parallels and meridians, except at the equator and the central meridian. Since the projection is symmetric the Tissot indicatrices are also symmetric.

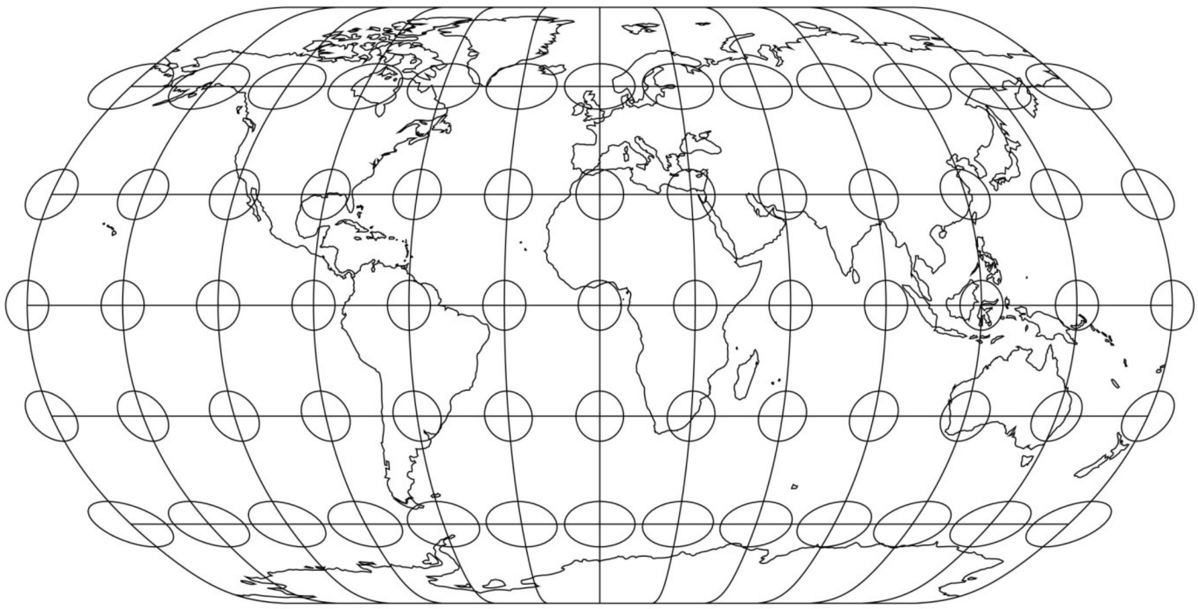


Figure 3: Tissot's indicatrices for the polynomial Natural Earth projection.

Tissot's indicatrices are not the only way to present the projection's characteristics. Instead of showing them on the map, different local measures of one particular distortion parameter, such as maximum areal distortion or maximum angular distortion, can be mapped by lines of equal distortion, called isocols (Figure 4 and 5).

Figure 4 presents isocols of areal distortion for the Natural Earth projection. The increased density of isocols at high latitudes also indicates an enlargement of those areas. Along the equator the areal distortion equals 0.87699. This value of areal distortion is the smallest for the Natural Earth projection. In Figure 4 the last isoline with the value of 5.0 can hardly be seen, as it is very close to the pole line. Areal distortion increases with latitude and does not change with longitude. All isocols of areal distortion are therefore parallel to the equator. Areal distortion is computed with $\sigma = a_j \cdot b_j$ with a_j and b_j the scale factors along the principal directions at position j on the sphere.

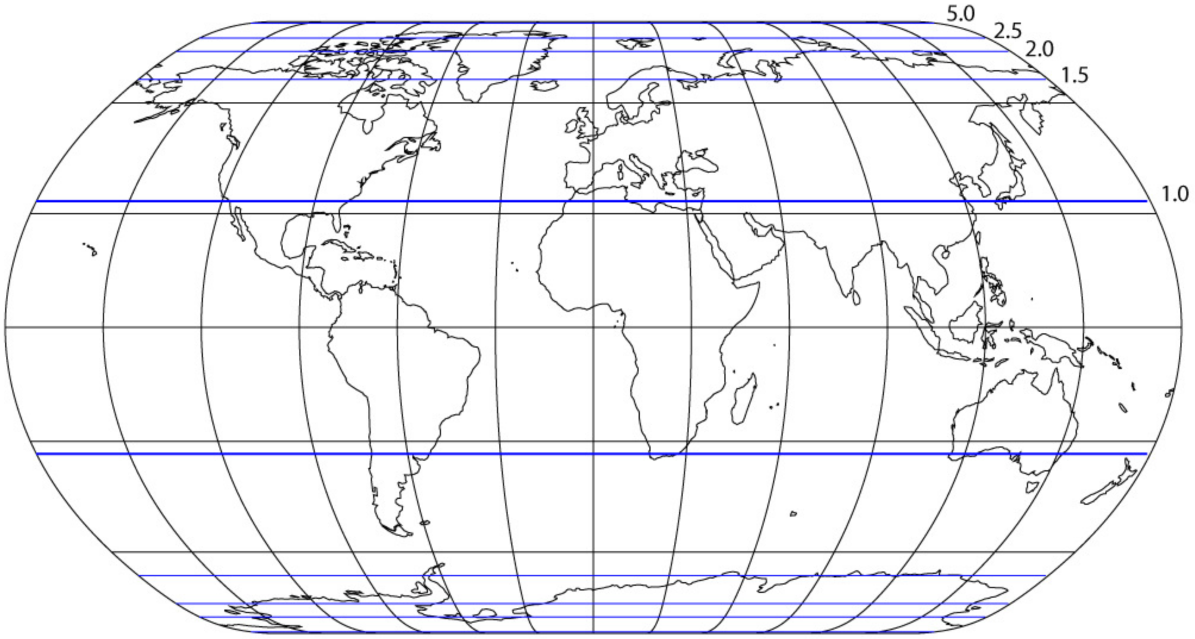


Figure 4: Isocols of areal distortion for the Natural Earth projection.

In Figure 5, the isocols of maximum angular distortion are presented. Their general pattern is common to all pseudo-cylindrical projections. Angular distortion is moderately near the equator and increases towards the edges of the graticule. It grows with an increased latitude. The last shown isoline is at 120 degrees, at the pole line, the distortion is 180 degrees. As for Tissot's indicatrices, the isocols of areal and maximum angular distortion are also symmetric.

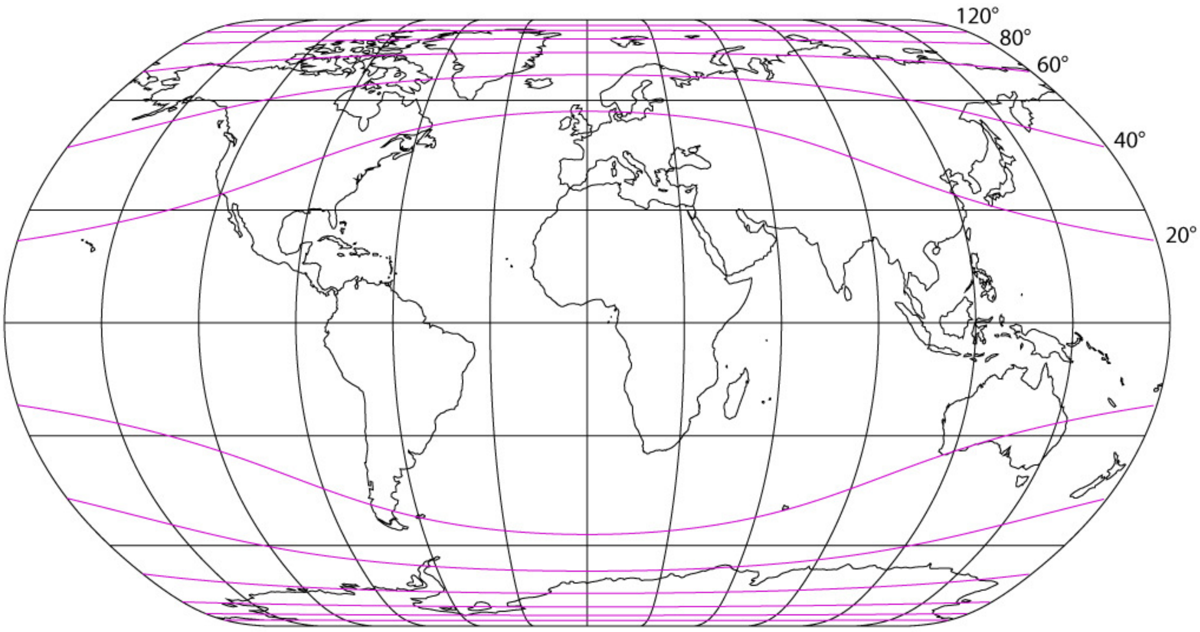


Figure 5: Isocols of maximum angular distortion for the Natural Earth projection.

2.3 The Robinson projection

There are considerable similarities between the Natural Earth projection and the Robinson projection. The Robinson projection is a pseudo-cylindrical projection, neither conformal nor equal area. The graticule is symmetric about the central meridian and the equator. The equator is 0.8487 times longer than the circumference of a sphere of equal area. The central meridian is a straight line 0.5072 longer than the equator. Other meridians are equally spaced elliptical arcs, and concave toward the central meridian (Robinson, 1974, Ipbüker, 2004).

The Robinson projection was designed by Arthur H. Robinson in 1974 for the Rand McNally Company. Robinson and Patterson used an identical approach for the design of their true pseudo-cylindrical projections. Both defined their projection by adjusting the appearance of the projected five-degree graticule in an iterative process – Robinson sketching the graticule with pen and paper, and Patterson fine-tuning it in Flex Projector. The identical procedure results in the same problems: both projections do not have normal analytical expressions.

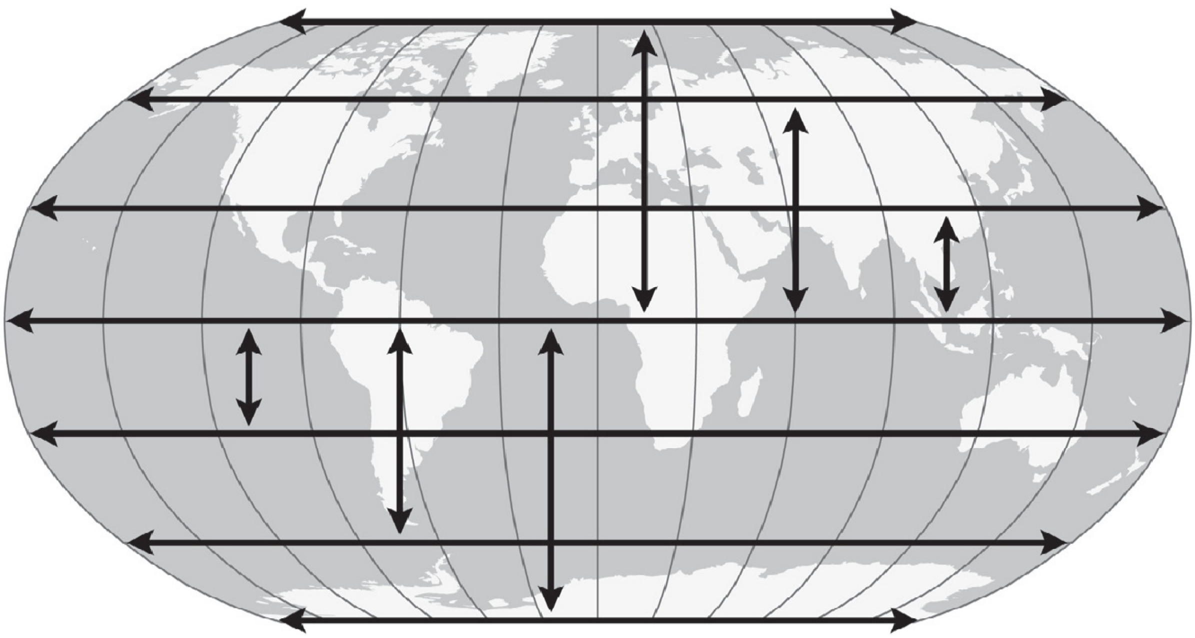


Figure 6: The two sets of tabular parameters define the Robinson projection. They are shown here with horizontal arrows (length of parallels) and vertical arrows (distance of parallels from the equator) (Jenny et al., 2008, page 13).

2.4 Analytical expressions for the Robinson projection

In the past, various authors tackled the problem of finding an analytical expression for the Robinson projection. Since the two projections are closely related, this section reviews existing mathematical models of the Robinson projection.

There are two general approaches to the mathematical modeling of graphically defined projections: (1) interpolation and (2) approximation. Both approaches have been applied to Robinson's projection.

Interpolating methods use a function that passes exactly through the reference points. Ipbüker (2004; 2005) presents a method based on multiquadric interpolation for the forward and the inverse projection. Others use interpolating methods for finding continuous expressions of l_φ and d_φ in Equation 1. For example, Snyder (1990) applies the central-difference formula by Stirling; Ratner (1991), Bretterbauer (1994), and Evenden (2008) use cubic spline interpolation (which is also used in Flex Projector); and Richardson (1989) reports that Robinson applied the Aitken interpolation scheme. A disadvantage of the mentioned interpolating methods is the large number of parameters required (more than 40 for the Robinson projection), and their relatively difficult implementation. For this reason they are not explored further here.

Approximating curves with parametric expressions that do not exactly replicate the original projection are an acceptable alternative, if deviations to the approximated values are small. Canters and Decleir (1989) present two polynomial equations (Equation 2) for approximating the Robinson projection. For the X coordinates they use even powers up to the order four, and for the Y coordinates odd powers up to the order five. Each expression contains three coefficients, and the constants k, s and π of Equation 1 are integrated with l_φ and d_φ. Their solution contains only six parameters, and is fast and simple to compute.

$$X = R \cdot \lambda \cdot (A_0 + A_2 \cdot \varphi^2 + A_4 \cdot \varphi^4)\ , \qquad \text{(Equation 2)}$$
$$Y = R \cdot (A_1 \cdot \varphi + A_3 \cdot \varphi^3 + A_5 \cdot \varphi^5),$$

where:

X and Y are the projected coordinates,

φ and λ are the latitude and longitude in radians,

R is the radius of the generating globe, with the coefficients being:

$A_0 = 0.8507$, $A_2 = -0.1450$, $A_4 = -0.0104$,

$A_1 = 0.9642$, $A_3 = -0.0013$ and $A_5 = -0.0129$.

A similar approach is proposed by Beineke (1991; 1995). For l_φ he suggests a polynomial with even degrees up to the sixth order, and for d_φ he proposes an exponential approximation with a real number exponent (Equation 3) (Beineke, 1991). This approach uses a total of eight parameters to approximate the Robinson projection. However, evaluating an exponential function with a real number exponent is slow. A test with the Java programming language, for example, shows that Beineke's exponential approximation is more than ten times slower to evaluate than a polynomial, such as the one by Canters and Decleir.

$$X = (d + e \cdot \varphi^2 + f \cdot \varphi^4 + g \cdot \varphi^6) \frac{\lambda}{\pi} \qquad \text{(Equation 3)}$$

$$Y = a \cdot \varphi + b \cdot s \cdot |\varphi|^c$$

where:

X and Y are the projected coordinates,

φ and λ are the latitude and longitude in radians,

with the coefficients being:

$a = 0.96047$, $b = -0.00857$, $c = 6.41$,

$d = 2.6666$, $e = -0.367$, $f = -0.150$, $g = 0.0379$ and

$$s = \begin{cases} 1 & \text{for } \varphi > 0 \\ -1 & \text{for } \varphi < 0 \end{cases}.$$

The approximating curves by Canters and Decleir, as well as Beineke, use a smaller number of parameters, and they are considerably simpler to program than the interpolating methods. Polynomial equations are the best in terms of computation speed and code simplicity, but higher-order terms might be necessary to minimize deviations from the original curve. However, polynomial approximations sometimes suffer from undulations if the maximum degree is too high, which must be avoided for a graticule to appear smooth. A drawback of polynomial equations is the difficulty of finding inverse equations that transform from projected X/Y coordinates to spherical coordinates. Indeed, an analytical inverse does not generally exist for higher-order polynomial equations. In order to transform Cartesian coordinates in to spherical coordinates, numerical approximation methods are necessary, such as the bisection or the Newton-Raphson root finding algorithm.

3 APPLIED NUMERICAL METHODS

This chapter presents two approximation methods using least squares adjustment (LSA). The first method is LSA of indirect observations and the second approach is a modified LSA of indirect observations with functionally dependent parameters presented by Mikhail and Ackerman (1976). LSA of indirect observations is used to determine the degree of the polynomial. The value of residuals and the reference variance are applied to estimate the results between different approximations. Similar to all approximation methods, this procedure does not provide a curve passing exactly through all points. Using the second method, parameter constraints are added in the functional model of LSA, considering geometric characteristics that are to be preserved. With this method, the values of parameters were calculated. The applied functional models in LSA procedure are presented in Chapter 4. The last section in this chapter presents the Newton-Raphson method used for inverting the polynomial projection.

3.1 Least squares adjustment of indirect observations

An adjustment of indirect observations is one of the two classical cases of LSA. The adjustment is performed with parameters and observations but with the restriction that only one observation is used in each condition equation. Therefore, the number of condition equations is the same as the number of observations (n). In each condition equation the observation is expressed with parameters (u). The general linear functional equations in this matrix form are:

$$\boldsymbol{v} + \boldsymbol{B} \cdot \boldsymbol{\Delta} = (-\boldsymbol{l} + \boldsymbol{d}) = \boldsymbol{f} \quad \text{or} \quad \boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{l} + \boldsymbol{v}, \qquad \text{(Equation 4)}$$

where:

$\boldsymbol{l}$ is the $n \times 1$ vector of observations,
$\boldsymbol{v}$ is the $n \times 1$ vector of residuals,
$\boldsymbol{\Delta}$ and $\boldsymbol{x}$ are the $u \times 1$ vectors of u parameters,
$\boldsymbol{B}$ and $\boldsymbol{A}$ are the $n \times u$ coefficient matrices of rank $= u$, and
$\boldsymbol{d}$ and $\boldsymbol{f}$ are $n \times 1$ vectors of constants.

Both matrix forms in Equation 4 present an identical mathematical model, and for the functional model used for the Natural Earth projection they provide the same results. The second form is more widespread and applied in this case. It is therefore presented here.

Usually the number of condition equations is larger than the numbers of parameters ($u < n$). For this reason there is some redundancy or degree of freedom (r), and no unique solution exists for the system in Equation 4. A unique least squares solution is obtained by adding the basic criterion: $\Phi = \boldsymbol{v}^T \cdot \boldsymbol{W} \cdot \boldsymbol{v} \rightarrow$ minimum, where $\boldsymbol{W}$ presents the weight matrix of the observations. With this criterion, the solution is given in Equation 5.

$$\boldsymbol{x} = \boldsymbol{N}^{-1} \cdot \boldsymbol{t} = \left(\boldsymbol{A}^{\boldsymbol{T}} \cdot \boldsymbol{W} \cdot \boldsymbol{A}\right)^{-1} \cdot \boldsymbol{A}^{\boldsymbol{T}} \cdot \boldsymbol{W} \cdot \boldsymbol{l}, \qquad \text{(Equation 5)}$$
$$\boldsymbol{v} = \boldsymbol{A} \cdot \boldsymbol{x} - \boldsymbol{l},$$

where:

$\boldsymbol{N}$ is the $u \times u$ reduced normal coefficient matrix, and

$\boldsymbol{t}$ is the $u \times 1$ reduced normal equations constant vector.

If LSA uses linear functions, no initial guesses are required. However, some conditions can only be expressed in non-linear manner. The parameters in the non-linear case are: $\widehat{\boldsymbol{x}} = \boldsymbol{x}^0 + \boldsymbol{x}$, where $\boldsymbol{x}^0$ is a vector of approximations and $\boldsymbol{x}$ contains only the corrections of these approximations. The coefficient matrix $\boldsymbol{A}$ contains partial derivatives of the condition equations with respect to all parameters. The partial derivatives are computed from initial guesses in vector $\boldsymbol{x}^0$. In the linear case, these partial derivatives are coefficients. To express non-linear conditions, certain means of linearization are used; e.g. Taylor's series (using the zero and first order terms and neglecting others). With the non-linear expression, Equation 5 is an iterative procedure where corrections of parameters are iteratively improved until they fulfill the required accuracy.

$$\hat{\sigma}_0^2 = \frac{\boldsymbol{v}^T \cdot \boldsymbol{W} \cdot \boldsymbol{v}}{r} \qquad \text{(Equation 6)}$$

Equation 6 presents an unbiased estimate of the reference variance, computed from the quadratic form of the residuals, where r is the redundancy. For the LSA of indirect observations, the redundancy is defined as a difference between the number of condition equations (n) and the number of unknown parameters (u): $r = n - u$.

3.2 Least squares adjustment of indirect observations with additional constraints

In the functional model of Equation 4, constraint equations can be added and defined only by parameters (Equation 7). This new group of extra equations implies that some of the parameters are functionally dependent. The parameter constraints are used when some of them must fulfill certain additional relationships from either a geometric or physical characteristic of the functional model. Since the constraint equations are expressed with the parameters from the adjustment, the number of dependent parameters (p) or their equations must be lower than the number of parameters (u): $p < u$. With the addition of constraint equations the redundancy is changed (Mikhail and Ackerman, 1976). For the LSA of indirect observations, the sum of conditions and constraints is equal to the redundancy plus the total numbers of all dependant and independent parameters: $n + p = r + u$.

$$\boldsymbol{C} \cdot \boldsymbol{x} = \boldsymbol{g} \qquad \text{(Equation 7)}$$

where:

$\boldsymbol{C}$ is the $p \times u$ coefficient matrix of rank $= p$ and

$\boldsymbol{g}$ is $p \times 1$ vector of constants.

Equation 7 expresses the relationship between the functionally dependent parameters that must be included in LSA (Equation 5). This step is presented in Equation 8, where the two systems in Equations 4 and 7 are solved together. The first row in Equation 8 is a normal LSA of indirect observations. The results are parameters without the constraints. In the second row of Equation 8, the corrections $\boldsymbol{\delta x}$ for the parameters are calculated. In the third row, the vector $\boldsymbol{x}$ is computed containing the coefficients of the polynomial approximation. And finally, the vector of residuals $\boldsymbol{v}$ is computed. The polynomial in $\boldsymbol{x}$ fulfills all additional parameter constraints expressed in Equation 7, and minimizes the correction of all observations.

$$\boldsymbol{l} + \boldsymbol{v} = \boldsymbol{A} \cdot \boldsymbol{x} \qquad \boldsymbol{N} = \boldsymbol{A}^T \cdot \boldsymbol{W} \cdot \boldsymbol{A} \qquad \boldsymbol{x}_0 = \boldsymbol{N}^{-1} \cdot \boldsymbol{A}^T \cdot \boldsymbol{W} \cdot \boldsymbol{l} \qquad \text{(Equation 8)}$$

$$\boldsymbol{C} \cdot \boldsymbol{x} = \boldsymbol{g} \qquad \boldsymbol{M} = \boldsymbol{C} \cdot \boldsymbol{N}^{-1} \cdot \boldsymbol{C}^T \qquad \boldsymbol{\delta x} = \boldsymbol{N}^{-1} \cdot \boldsymbol{C}^T \cdot \boldsymbol{M}^{-1} \cdot (\boldsymbol{g} - \boldsymbol{C} \cdot \boldsymbol{x}_0)$$

$$\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{\delta x} \qquad \boldsymbol{v} = \boldsymbol{A} \cdot \boldsymbol{x} - \boldsymbol{l}$$

As the conditions are linear, no iterations are needed to solve the functional model, and no initial guesses are required for the unknown parameters. All constraints can be expressed with functionally dependent parameters. Even if the conditional equations are linear, non-linear constraints can be used. In this case, matrix $\boldsymbol{C}$ contains partial derivatives of the constraint equations with respect to all parameters in vector $\boldsymbol{x}$, calculated from parameter values in vector $\boldsymbol{x}_0$. However, non-linear constraints can only partially be fulfilled. If the conditions are originally non-linear, constraints are expressed with the corrections of initial guesses for parameters. The form of those constraints depends on the values of the initial guesses.

3.3 The Newton-Raphson iteration method

Many cartographic projections do not have an equation for the inverse projection from projected coordinates X/Y to spherical ones φ/λ. For the inverse projection different approximation and iteration methods are used, such as the bisection method, Euler method or the Newton-Raphson root finding algorithm.

The Newton-Raphson algorithm is a numerical method for finding successively better approximations to the roots or zeroes of a real-valued function (Wikipedia), and is commonly used for the numerical calculation of non-linear equations. Equation 9 shows the general form of the Newton-Raphson algorithm.

$$x_n = x_{n-1} - F(x_{n-1}) \cdot (F'(x_{n-1}))^{-1} \rightarrow F(x_{n-1}) = 0, \qquad \text{(Equation 9)}$$

where:
$F(x_{n-1})$ and $F'(x_{n-1})$ are a given function and its derivative,
x_{n-1} and x_n are the previous and the next solution of the given function, and
n and $(n-1)$ the steps of the iterative process.

The algorithm starts with the first initial guess x_0 and computes an improved approximation of the solution: x_1. In the next step, this new approximation is used instead of x_0, and x_2 is again closer to the solution. The procedure continues until the difference between approximated values is sufficiently small, ε. The iteration stops at x_n, which is the final solution. When the first initial guess is close enough to the final solution and the derivative of the given function has the same sign (just positive or just negative, but not zero) as in the point of the solution, the algorithm converges very quickly and its convergence is quadratic.

Geometrically, this method does not search the intersection of the graph with the x-axis, but the intersection point x_n of the tangent line to the graph with the x-axis in the start point $(x_0, F(x_0))$. From this geometrical meaning the alternative name for this method is tangent iteration method (Bronštejn et al., 1974).

Figure 7 demonstrates this method. As an initial guess the value x_1 is used, from which the tangent and intersection of the tangent line with the x-axis x_2 is computed. In step B, an approximation of the root, x_2 is used, and an improved approximation is computed. The final result in step D is x_5.

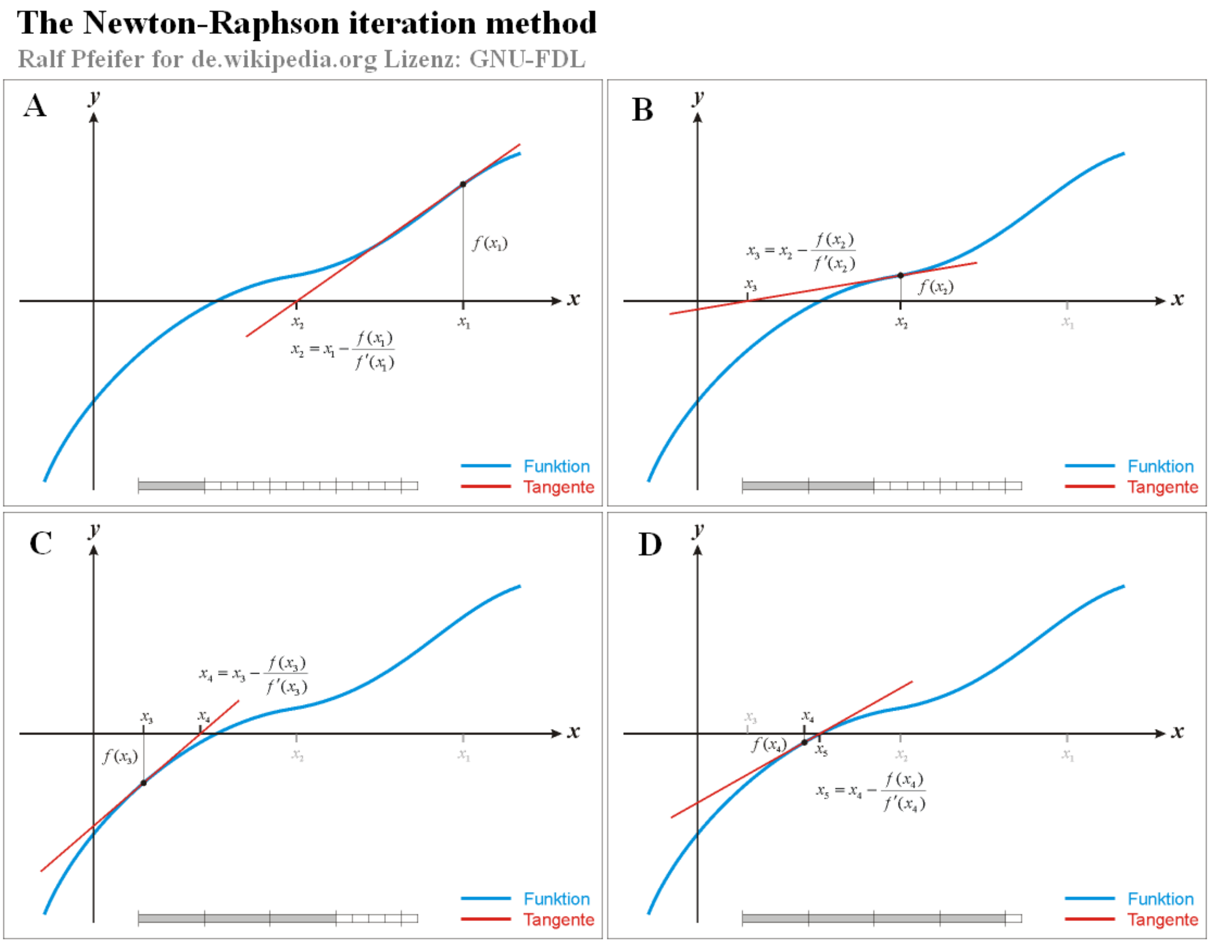


Figure 7: Pictures A, B, C and D present separated phases of the Newton-Raphson iteration method. The final solution is x_5 (Wikipedia).

4 A POLYNOMIAL APPROXIMATION OF THE NATURAL EARTH PROJECTION

In a trial-and-error process, a polynomial approximation with a minimum number of terms was determined for the original Natural Earth projection. Polynomials of varying degrees and different number of terms were selected and their coefficients were computed using the method of least squares with constraints. Two criteria were used to evaluate variants developed with this iterative trial-and-error procedure. First, the number of polynomial terms and the number of multiplications required to evaluate the equation need to be minimized. This criterion is important for simplifying the programming of the equations. It is also relevant for accelerating computations, for example, for web mapping application that projects maps on the fly using JavaScript or other interpreted programming languages that are comparatively slow. The second criterion aims at minimizing the absolute differences between the original projection and the approximated projection. Differences should be minimal throughout the entire projection.

When designing the original Natural Earth projection, special focus was given to the smoothness of the rounded corners where the meridians meet the horizontal pole lines. It was found that the graphical tools and the cubic spline interpolation in Flex Projector do not provide sufficient control for defining rounded corners with adequate smoothness. The development of a polynomial approximation provided the possibility to further improve this distinguishing characteristic of the Natural Earth projection, which was also a wish of the designer of the original projection. The new polynomial form of the projection therefore deliberately deviates from the original projection by adding curvature to the corners. It has to be noted that the changes applied to the smoothness of the corners are entirely esthetic and satisfy the authors' sensibilities. They result in a subjective improvement that cannot be evaluated with objective criteria. Furthermore, changes are not applied to improve the projection's distortion characteristics.

The development of the polynomial approximation was assessed by Tom Patterson, the projection designer. Since his wish was to improve the rounded corners, his graphical evaluation has more significance than the absolute differences between the original projection and the approximated projection. Approximations with different polynomial degrees were evaluated during this process to minimize the absolute difference between the approximated and the original projection. For each approximation the reference variance was computed from the residuals. When the variance did not change much with an increased polynomial degree, the approximation was considered sufficient. The polynomial equations were then implemented in Flex Projector for graphical review of the results and also for the comparison with the original projection. This graphical evaluation was made by the designer. By using this graphic approach for the evaluation of the new projection, the results reflect the subjective evaluation of the designer. The polynomial approximation of the Natural Earth projection (hereafter the term polynomial Natural Earth projection is used for this approximation) was developed in six separated phases, presented in the following subsections. Appendix A provides the Matlab code, which is used for the evaluation of the polynomial coefficients.

4.1 Derivation of the polynomial to express Cartesian coordinates

Each Cartesian coordinate, X and Y, can be expressed by two spherical coordinates, φ and λ. The general form of a polynomial with two variables (Equation 10) has $\frac{1}{2}(n+1)(n+2)$ terms, where n is the degree of the polynomial. In the case of 3 degrees, 20 parameters are needed to express both coordinates on the map plane. Some values of those parameters are equal or close to zero because of the characteristics of the projection. Since these terms are not needed, the general form can be simplified.

$$p(\varphi,\lambda) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} c_{ij}\lambda^i\varphi^j \qquad \text{(Equation 10)}$$

To simplify the terms in Equation 10, the following considerations given by Canters (2002) for deriving new graticules with polynomials were taken into account:

(1) The Natural Earth projection is symmetric about the X and Y-axis;

(2) It has straight but not equally spaced parallels;

(3) The parallels are equally divided by meridians.

The equator of the graticule represents the X-axis and the central meridian the Y-axis (Figure P. 2, on page 38). Longitude λ changes with the X coordinate. The Y coordinate has the same sing as latitude φ. When the projection is symmetric about the Y-axis, even powers of λ preserve the sign of the Y coordinate as provided by latitude φ, and odd powers of λ preserve the sign for the X coordinate. Symmetry about the X-axis is obtained in a similar way. Even powers of φ preserve the sign of the X coordinate as provided by longitude λ and odd powers of φ preserve the sing for the Y coordinate.

By taking the symmetry relative to both axes into consideration, most of the terms in Equation 10 are eliminated. For the X coordinate, all coefficients of even powers of λ and odd powers of φ are removed, and for the Y coordinate all terms with odd powers of λ and all even powers of φ are removed. Equation 11 presents this removal for the polynomial of 3rd degree (Canters, 2002).

(Equation 11)

$$X = \cancel{A_{00}} + \cancel{A_{01}\varphi} + \cancel{A_{02}\varphi^2} + \cancel{A_{03}\varphi^3} + A_{10}\lambda + \cancel{A_{11}\lambda\varphi} + A_{12}\lambda\varphi^2 + \cancel{A_{20}\lambda^2} + \cancel{A_{21}\lambda^2\varphi} + A_{30}\lambda^3 + \cdots$$
$$Y = \cancel{B_{00}} + B_{01}\varphi + \cancel{B_{02}\varphi^2} + B_{03}\varphi^3 + \cancel{B_{10}\lambda} + \cancel{B_{11}\lambda\varphi} + \cancel{B_{12}\lambda\varphi^2} + \cancel{B_{20}\lambda^2} + B_{21}\lambda^2\varphi + \cancel{B_{30}\lambda^3} + \cdots$$

The second characteristic, straight but not equally spaced parallels, allows for making the Y coordinate a function of latitude only, since the parallels do not change with longitude. The unequal distribution of parallels of the Natural Earth projection has to be expressed as a non-linear function of latitude (Equation 12). A linear expression would result in equally spaced parallels (Canters, 2002).

The last mentioned characteristic, the maintenance of the equal spacing of meridians, forms the function for X coordinates. This characteristic is dependent on longitude, and for true pseudo-cylindrical projection a linear expression should be used (Canters, 2002). Therefore the expression of X coordinate contains only the terms with linear longitude (Equation 12).

$$X = A_{10}\lambda + A_{12}\lambda\varphi^2 + \cancel{A_{30}\lambda^3} + \cdots = \lambda \cdot (A_{10} + A_{12}\varphi^2 + \cdots) \qquad \text{(Equation 12)}$$
$$Y = B_{01}\varphi + B_{03}\varphi^3 + \cancel{B_{21}\lambda^2\varphi} + \cdots = B_{01}\varphi + B_{03}\varphi^3 + \cdots$$

As can be seen in Equation 12, for the polynomial of third degree only 4 coefficients remain, which is much less than the 20 terms in Equation 11. If these parameters were not removed from the expression, LSA would determine them as equal or close to zero. Therefore in the next phases for the X coordinate only even powers of latitude φ, multiplied with the longitude λ are used, and Y coordinate contains only odd powers of latitude φ.

4.2 Increasing the order of the polynomial

Once the form of the polynomials is determined, finding an appropriate degree is the next step. If the degree is too low, the curve deviates from the control points with big residuals. If the degree is too high, the curve waves through these points, although the residuals are relatively smaller. Different variants were compared on the basis of the reference variance, calculated from the quadratic form of residuals, and by a graphical comparison of the approximated and the original curves (Figure 8). A low polynomial degree makes the approximate curve more dissimilar to the original one. For example, in Figure 8 the curve of degree four deviates from the original because the degree is too low. With a higher degree (degree twelve in Figure 8) the curve is closer to the original and the reference variance becomes smaller. To achieve a better curve fitting for the Natural Earth projection, the degree of polynomials in Equation 12 is increased to 12 (Equation 13).

$$X = \lambda \cdot \left(A_{1,0} + A_{1,2}\varphi^2 + A_{1,4}\varphi^4 + A_{1,6}\varphi^6 + A_{1,8}\varphi^8 + A_{1,10}\varphi^{10} + A_{1,12}\varphi^{12} + \cdots\right) \qquad \text{(Equation 13)}$$
$$Y = B_{0,1}\varphi + B_{0,3}\varphi^3 + B_{0,5}\varphi^5 + B_{0,7}\varphi^7 + B_{0,9}\varphi^9 + B_{0,11}\varphi^{11} + \cdots$$

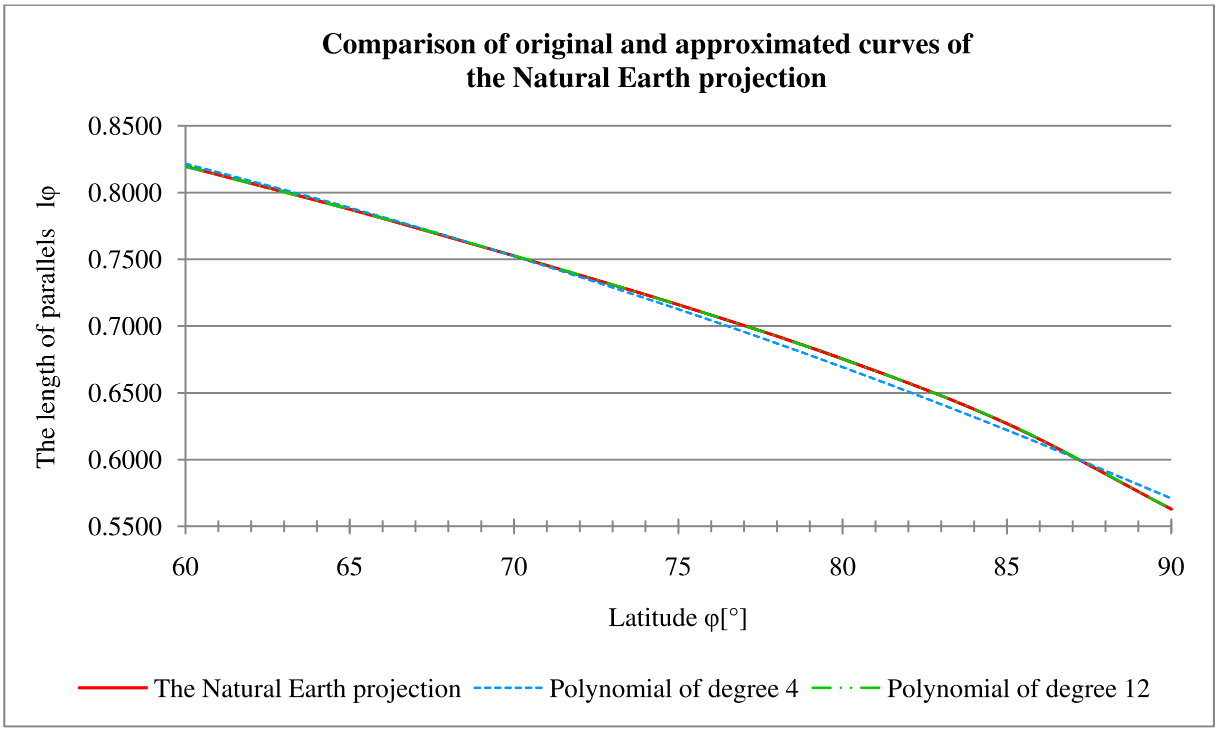


Figure 8: Fitting curves of different polynomial degrees. Canters and Declair's polynomial (4th degree) clearly deviates from the polynomial of degree 12.

In Table 2 the effect of increasing the order of the polynomial is presented. For each tabular parameter different polynomial degrees were used in LSA, with the reference variance (Equation 6) and the maximal residual used for the first evaluation. The length of the parallels and their distribution were approximated separately, and they are presented in Table 2. The degree of polynomials in the functional model was increased by one term in each step. Since the reference variance and the maximal residuals of the 10th and 12th degree do not change much, the curve could start undulating between control points. The maximal residuals were about 1 millimeter at a scale of 1: 5,000,000. This scale presents a map with dimensions of 3.489 × 1.814 meters. The conclusion was that this solution is accurate enough for the graphical evaluation. The polynomial functions were implemented in the Flex Projector application. The used functional model is presented in the next sub-subsection.

Table 2: Reference variances and maximal residuals of LSA at different polynomial degrees. The last line shows the values of the LSA after the removal of the terms with small contribution. Note: Units are millimeters on the map at a scale 1 : 5,000,000, and the radius of the generating globe equals 6,378,137 meters.

Polynomial degree	The length of parallels (l_φ)		The distance of parallels (d_φ)	
	$\hat{\sigma}_0^2$ [mm]	max (v) [mm]	$\hat{\sigma}_0^2$ [mm]	max (v) [mm]
6th	7.322	14.701	2.564	5.483
8th	2.328	4.556	1.323	2.447
10th	0.494	1.186	0.499	0.760
12th	0.406	1.173	0.291	0.813
14 th	0.359	0.911	0.186	0.476
*12 th	0.406	1.081	0.287	0.823

4.2.1 The functional model for least squares adjustment

To determine the desired polynomial orders, LSA of indirect observations was used for computing the necessary residuals for evaluation. For each LSA the functional model must be composed first. The original Natural Earth projection is defined by 37 control points distributed over the complete lateral meridian (with $\varphi \in [-\pi/2, \pi/2]$ and a control point every 5 degrees). The functional model of the LSA is given in Equation 14, which is derived from Equations 1 and 13:

$$A_1 + A_2\varphi_i^{\,2} + A_3\varphi_i^{\,4} + A_4\varphi_i^{\,6} + \cdots = s \cdot l_\varphi, \qquad (\text{for } i,j = 1, \dots 37)$$

$$B_1\varphi_j + B_2\varphi_j^{\,3} + B_3\varphi_j^{\,5} + B_4\varphi_j^{\,7} + \cdots = s \cdot d_\varphi \cdot k \cdot \pi. \qquad (\text{Equation 14})$$

The coefficients of the two polynomial expressions in Equation 14 are unknown parameters, and the polynomial powers of latitude are known coefficients for the LSA. Both systems of equations were adjusted separately (Equation 14), with n denoting the number of rows in the coefficients matrix $\boldsymbol{A}$ ($n = 37$), and u the number of parameters which is increased. The separate matrix or vector terms to express the length of parallels (l_φ) and the vertical distance of the parallel from the equator (d_φ) are introduced in Equations 15 and 16.

$$\boldsymbol{l_1} = \begin{bmatrix} s \cdot l_{\varphi,01} \\ s \cdot l_{\varphi,02} \\ \vdots \\ s \cdot l_{\varphi,37} \end{bmatrix} \qquad \boldsymbol{A_1} = \begin{bmatrix} 1 & \varphi_{01}^{\,2} & \varphi_{01}^{\,4} & \dots \\ 1 & \varphi_{02}^{\,2} & \varphi_{02}^{\,4} & \dots \\ \vdots & \vdots & \vdots & \dots \\ 1 & \varphi_{37}^{\,2} & \varphi_{37}^{\,4} & \dots \end{bmatrix} \qquad \boldsymbol{x_1} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \end{bmatrix} \qquad (\text{Equation 15})$$

$$\boldsymbol{l_2} = \begin{bmatrix} s \cdot k \cdot d_{\varphi,01} \cdot \pi \\ s \cdot k \cdot d_{\varphi,02} \cdot \pi \\ \vdots \\ s \cdot k \cdot d_{\varphi,37} \cdot \pi \end{bmatrix} \qquad \boldsymbol{A_2} = \begin{bmatrix} \varphi_{01} & \varphi_{01}^{\,3} & \varphi_{01}^{\,5} & \dots \\ \varphi_{02} & \varphi_{02}^{\,3} & \varphi_{02}^{\,5} & \dots \\ \vdots & \vdots & \vdots & \dots \\ \varphi_{37} & \varphi_{37}^{\,3} & \varphi_{37}^{\,5} & \dots \end{bmatrix} \qquad \boldsymbol{x_2} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \vdots \end{bmatrix} \qquad (\text{Equation 16})$$

The index 1 presents the approximation of l_φ, and index 2 the approximation of d_φ. Vectors $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ contain the parameters, *i.e.* the unknown polynomial coefficients. In this particular case, no parameters are included in the matrices $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$, which results in a linear system that can be solved using the method of least squares in Equation 5. The vectors $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ represent the minimized residuals after adjustments. $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ are used for evaluating the reference variance of the model, and the differences between the original and the approximated graticule. The vectors $\boldsymbol{l_1}$ and $\boldsymbol{l_2}$ include l_φ or d_φ, multiplied by the constant factors s, k and π as in Equation 14. Since this is a linear model, no initial guess is required for the parameters. All observations in vectors $\boldsymbol{l_1}$ and $\boldsymbol{l_2}$ have the same relevance. Therefore the weight matrices ($\boldsymbol{W_1}$ and $\boldsymbol{W_2}$) are equal to the identity matrices in both cases. The weight matrix can be removed from Equations 5, 6 and 8, and calculations are simplified.

4.3 Removing the terms with small contributions

Reducing the number of terms in the polynomials accelerates computations. The values of the polynomial coefficient indicate a possibility to remove some of the polynomial terms.

The results of a LSA of polynomials with 12^{th} degree indicate a possibility to remove the sixth and eighth power terms in the equation that expresses X coordinates. As for the equation of Y coordinate, the fifth power term has the smallest value. All three terms were removed in this step, and LSA with the new functional model (without these terms) was computed. Removing the terms did not considerably change the graphical appearance. Moreover, Table 2 shows that reference variances and maximal residuals are even better. Therefore the removal is acceptable.

The equations by Robinson (Equation 1) can be expressed with these reduced polynomials. By multiplying the polynomials (Equation 13) with the radius of the generating globe, the polynomials express the multiplied tabular parameters l_φ and d_φ, including the constant factors s, k and π in Equation 17. This form of equations can be accepted as the final form, but coefficients included in polynomials must be determined with additional constraints.

$$X = R \cdot \lambda \cdot (A_1 + A_2\varphi^2 + A_3\varphi^4 + A_4\varphi^{10} + A_5\varphi^{12}) \qquad \text{(Equation 17)}$$
$$Y = R \cdot (B_1\varphi + B_2\varphi^3 + B_3\varphi^7 + B_4\varphi^9 + B_5\varphi^{11})$$

4.4 Adding constraints in the least squares adjustment

The graticule of the Natural Earth projection provided by polynomial equations is not exactly the same size as the original graticule. The deviations are caused by the LSA approximation method. Residuals from LSA present this discrepancy for each coordinate of the control points. In Figure 8 this discrepancy is evidently represented for the length of parallels. At 90 degrees of latitude this change can be seen by comparing the original Natural Earth projection with an approximated curve of the fourth polynomial degree. To preserve the size and proportions of the graticule the length of the equator and the distance between the pole line and the equator have to remain the same.

Hence, when estimating the polynomial coefficients with the method of least squares, two additional constraints were added to bring the polynomial graticule to the exact same size as the original graticule: (1) the length of the parallel l_φ at 0 degrees must be 1; (2) the relative distance between the equator and the parallel d_φ at 90 degrees must be 1. Both constraints are expressed with parameters, which is possible because the expressions in Equation 17 are linear.

For computing the polynomial coefficients of the relative length l_φ, 37 control points are used covering the whole range of possible latitude values between $-\pi/2$ and $+\pi/2$ with a distance of five degrees between each pair of control points. The symmetrical arrangement of the control points around the equator guarantees a continuously differentiable function.

The first additional constraint (1) for the length l_φ can be derived from Equations 14 and 17. This functional dependence is described in the following equation:

$$\varphi = 0 \rightarrow l_\varphi = 1 \rightarrow A_1 = s \qquad \text{(Equation 18)}$$

The first coefficient A_1 was forced to equal the value of the internal scale factor s (Equation 1), which ensures that the length of the equator remains the same. Since this constraint is the only one for the length l_φ, it must be written separately with the matrix $\boldsymbol{C_1}$ and vector $\boldsymbol{g_1}$ for Equation 7. With the matrix and vectors from Equation 15 it forms the functional model for the length of parallels.

$$\boldsymbol{C_1} = [1 \quad 0 \quad 0 \quad 0 \quad 0] \qquad \boldsymbol{g_1} = [s] \qquad \text{(Equation 19)}$$

The constraint (2) is applied to the distance d_φ. The fixed distance of the parallel at 90 degrees is expressed in a similar way as the constraint (1) in Equation 18, and it must also be present in a separated matrix form (Equation 21).

$$\varphi = \frac{\pi}{2} \rightarrow d_\varphi = 1 \rightarrow \qquad \text{(Equation 20)}$$
$$B_1 \cdot \frac{\pi}{2} + B_2 \left(\frac{\pi}{2}\right)^3 + B_3 \left(\frac{\pi}{2}\right)^7 + B_4 \left(\frac{\pi}{2}\right)^9 + B_5 \left(\frac{\pi}{2}\right)^{11} = \mathrm{s} \cdot \mathrm{k} \cdot \pi$$

$$\boldsymbol{C_2} = \left[\frac{\pi}{2} \quad \frac{\pi^3}{8} \quad \frac{\pi^7}{128} \quad \frac{\pi^9}{512} \quad \frac{\pi^{11}}{2048}\right] \qquad \boldsymbol{g_2} = [\mathrm{s} \cdot \mathrm{k} \cdot \pi] \qquad \text{(Equation 21)}$$

Matrices and vectors in the equations 15, 16, 19 and 21 form two functional models, whose results provide the polynomial graticule of the Natural Earth projection. For the calculation of polynomial coefficients, LSA of an indirect observation with additional constraints was used with Equation 8. The new polynomial coefficients approximate the original graticule, but the width and height are the same as for the original projection.

4.5 Improving the smoothness of the rounded corners

The distinguishing marks of the Natural Earth projection are the rounded corners where bounding meridians meet the pole lines. By decreasing the length of the pole lines by a small amount, the cured corners were added during the design process of the original projection, in order to serve five purposes (Jenny et al. 2007):

(1) The graticule appears more like a spherical Earth;
(2) With a shorter pole line, areal distortion at poles is reduced;
(3) Converging meridians are suggesting that the poles are in fact points;
(4) Curves convey a sense of classic elegance and bring a special esthetic look;
(5) Rounded corners are rare among pseudo-cylindrical projections – they are useful to distinguish between similar projections.

Two additional measures were required to increase the smoothness of the rounded corners between the meridians and the pole lines. For the function of Y coordinate, an additional constraint was added to the method of least squares, fixing the slope of the polynomial to 7 degrees at the poles. The second measure for improving the smoothness of the corners was taken by slightly reducing the length of the pole line from 0.563 to 0.550 before computing the polynomial coefficients. This change reduces a bulge, which was caused by fixing the slope. The result is a new polynomial Natural Earth projection that deliberately deviates from the original projection near the poles.

This section continues by providing further details on these two additional measures. Firstly, the implementation of this additional constraint in matrices of LSA is presented. Secondly, it is described how the corners can be changed. Thirdly, the usage of the chain rule and graphical explanation is given. Finally, the value of the slope is explained.

The constraint for fixing the slope of the polynomial to 7 degrees at the poles was added to the method of least squares. The slope of the polynomial, which expresses the distance of the parallels from the equator, has a large influence on the slope of meridians at the pole lines. For this reason a derivative of the Y coordinate function is used resulting in Equation 22, which adds one row to Equation 21:

$$\boldsymbol{C_2} = \begin{bmatrix} \frac{\pi}{2} & \frac{\pi^3}{8} & \frac{\pi^7}{128} & \frac{\pi^9}{512} & \frac{\pi^{11}}{2048} \\ 1 & \frac{3\pi^2}{4} & \frac{7\pi^6}{64} & \frac{9\pi^8}{256} & \frac{11\pi^{10}}{1024} \end{bmatrix} \qquad \boldsymbol{g_2} = \begin{bmatrix} \mathrm{s} \cdot \mathrm{k} \cdot \pi \\ \tan(7°) \end{bmatrix}$$ (Equation 22)

To change the corners, the slope of meridians must be minimized at 90 degress. Each meridian is a graph of two variables X and Y. A meridian is a function of the Y coordinate, where X is a parameter: $Y = f(X)$. When the curve approaches the pole line, the slope decreases, and at the end the final slope should equal zero (Equation 23) to achieve completely smooth corners. However, the meridians are not expressed with this function, but each X and Y coordinate is expressed by spherical coordinates as a parametric curve.

To solve this issue, the chain rule was used to find out which polynomial has the main influence on the graticule. The derivative of the Y function with respect to its only parameter (latitude φ) was extended by the chain rule in Equation 23. Using this formula for computing the derivatives, which are composed of two or more functions, the connection among all applicable derivatives is made. The partial derivative of X coordinate with respect to the longitude λ is neglected here and the longitude is used as a constant value ($\lambda = 180°$).

$$\frac{dY}{d\varphi} = \frac{dY}{dX} \cdot \frac{dX}{d\varphi}$$ (Equation 23)

From Equation 23 it can be seen that the slope of the meridians at 90 degrees (when it is equal to zero) directly changes the slope of Y function. It is obvious that the slope of the polynomial which defines the Y-coordinate should equal zero (close to zero) to achieve completely smooth corners. In this case, the slope of the polynomial, which expresses the distance of the parallels from the equator, has the main influence on the slope of meridians at pole lines. The described connection between these two slopes is also indicated by the graphs of both tabular parameters (Figure 9).

The graph showing the length of the parallels (Figure 9, left) at 90 degrees is decreasing and the slope of this curve is the largest at this point. On the other hand, the slope of the second tabular parameter (Figure 9, right) is slowly decreasing towards 90 degress. Therefore, a change of the slope to the first curve has a larger effect on the graticule than a change to the slope of the Y function.

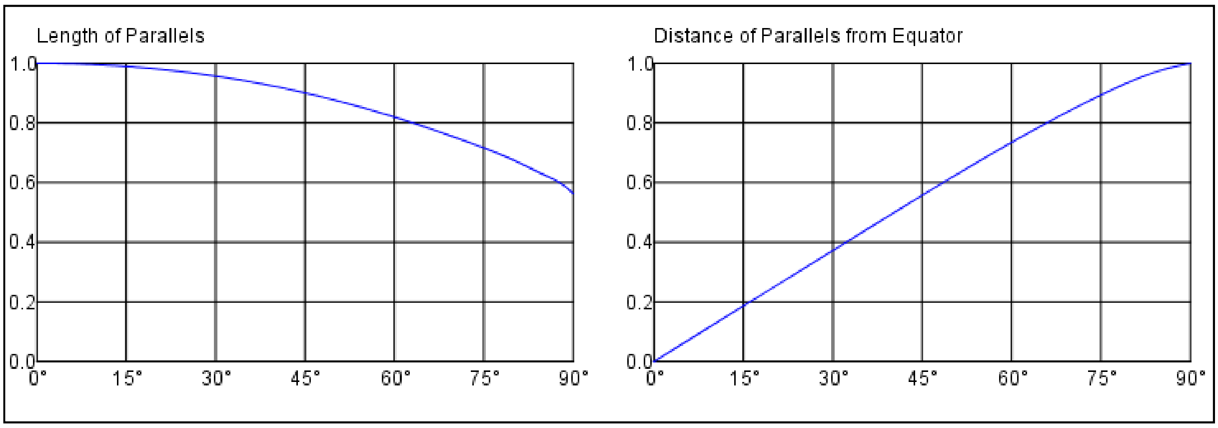


Figure 9: Screenshots of graphs of both tabular parameters with dependence of the latitude in Flex Projector. Abscissa has latitude values φ in degrees units, and ordinate contains the values of parameters (without units).

For perfect smoothness at the corners, the slope of the border meridians at pole lines should equal zero, but this additional constraint adds undesired bulges at the corners. To prevent this deviation, the slope at the poles from vector $\boldsymbol{g_2}$ in Equation 22 was increased to 7 degrees. With this small slope, corners are almost invisible, and bulges are reduced. To hide the small bulges in the corners, the length of the pole line was reduced. This approach does not provide perfect smoothness, but from a cartographer's point of view, the corners are not noticeable.

4.6 Inverting the polynomial equation

The inverse of a map projection transforms Cartesian coordinates into spherical coordinates. To determine the inverse of the polynomial Natural Earth projection, Equation 17 must be inverted, but analytical expressions of the inverse of polynomials do not exist. However, there are many methods for solving polynomial systems (Elkadi et al., 2005). For inverting the Natural Earth projection the Newton-Raphson root finding algorithm was chosen because it converges rapidly, it is easy to compute, and it requires only one initial guess.

The system defined by the two polynomials (Equation 17) has two known variables (the Cartesian coordinates X and Y) and two unknown ones (the spherical coordinates φ and λ). It is solved by finding the latitude φ in Y function and then solving X equation for the unknown longitude λ.

The function $F(x_n)$ from Equation 9 is formed by converting Equation 17 to Equation 24:

$$F(\varphi_n) = B_1\varphi_n + B_2{\varphi_n}^3 + B_3{\varphi_n}^7 + B_4{\varphi_n}^9 + B_5{\varphi_n}^{11} - Y \cdot R^{-1} = 0 \qquad \text{(Equation 24)}$$

The iterative approximation is repeated until a sufficiently accurate solution is reached. Convergence to the solution is quadratic for Equation 24, because the derivative $F'(\varphi_n)$ is positive for all $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, and $F(\varphi_n)$ has therefore no local minimum or maximum in the valid range of φ. The closest local extremum is at $\varphi = \pm 1.59$, which is outside the valid range of φ. The quotient $Y \cdot R^{-1}$ can be used as an initial guess for the Newton-Raphson algorithm, as it is in the range of the latitude φ, and does not have any local extremum in this range (Equation 25).

$$Y \cdot R^{-1} \in [-s \cdot k \cdot \pi, s \cdot k \cdot \pi] \subset \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad \text{(Equation 25)}$$

An alternative general method for inverting arbitrary map projections without explicit inverse expressions was described by Ipbüker and Bildirici (2002). They utilize the two forward expressions to calculate the geographical coordinates φ and λ using Jacobian matrices. For the Natural Earth projection, this method based on Jacobian matrices results in the same values as the Newton-Raphson approach presented here. For both methods, an equal number of iterations are required (with an identical ε). However, the Newton-Raphson method is faster as it involves fewer calculations, and it is algorithmically simpler.

5 ANALYTICAL EXPRESSION FOR THE NATURAL EARTH PROJECTION – RESULTS

The final result of this thesis is the forward polynomial expression of the Natural Earth projection and its inverse procedure. The equations improve the rounded corners of the graticule, which was also a goal of this study. In this chapter the forward polynomial expression with all polynomial coefficients is presented. It is the result of the first five phases from the previous section. The sixth phase provides the inverse procedure using the Newton-Raphson method, which is explained in the second part of this section. The last section discusses the difference between the original and the polynomial Natural Earth projections.

5.1 Forward polynomial expressions

The polynomial expression for the Natural Earth projection was already given in Equations 17 in section 4.3. The polynomials are of higher degrees than those by Canters and Decleir (1989) for the Robinson projection, as explained in Chapter 2.4. Higher degrees are required for the Natural Earth projection to smoothly model the curved corners connecting the meridian lines and the horizontal pole line.

Equation 26 replaces both l_φ and the factor s in Equation 1 with a polynomial expression (where $A_1 = s$). The polynomial in Equation 27 includes the constant factors s, k, π, and d_φ for reducing the number of required multiplications and accelerating calculations. In order to further accelerate computations, the number of polynomial terms in the least squares models has been reduced. Equation 26 has no terms with degrees six and eight, and Equation 27 has no term with degree five. Due to the characteristics presented in Chapter 4.1, Equation 26 contains only even powers of φ that are multiplied by longitude λ, and Equation 27 only consists of odd power terms of φ.

$$X = R \cdot \lambda \cdot (A_1 + A_2\varphi^2 + A_3\varphi^4 + A_4\varphi^{10} + A_5\varphi^{12}) \quad \text{(Equation 26)}$$

$$Y = R \cdot (B_1\varphi + B_2\varphi^3 + B_3\varphi^7 + B_4\varphi^9 + B_5\varphi^{11}) \quad \text{(Equation 27)}$$

where:

X and Y are the projected coordinates,

φ and λ are the latitude and longitude in radians,

R is the radius of the generating globe, and

A_1 to A_5 and B_1 to B_5 are coefficients given in Table 3.

Table 3: Coefficients for the polynomial expression of the Natural Earth projection.

Coefficients for Equation 26		Coefficients for Equation 27	
A_1	0.870700	B_1	1.007226
A_2	-0.131979	B_2	0.015085
A_3	-0.013791	B_3	-0.044475
A_4	0.003971	B_4	0.028874
A_5	-0.001529	B_5	-0.005916

The final polynomial coefficients are given in Table 3. They were determined with the Matlab code in Appendix A written for this purpose. Calculations in Matlab are made with a precision of 10^{-15}. To simplify the implementation, the parameters given in Table 3 are rounded to six decimal places. Rounding the parameters results in deviations smaller than 0.02 mm at a scale of 1 : 5,000,000. Parameters with only five decimal places would result in larger deviations (close to 2 mm) at the same scale.

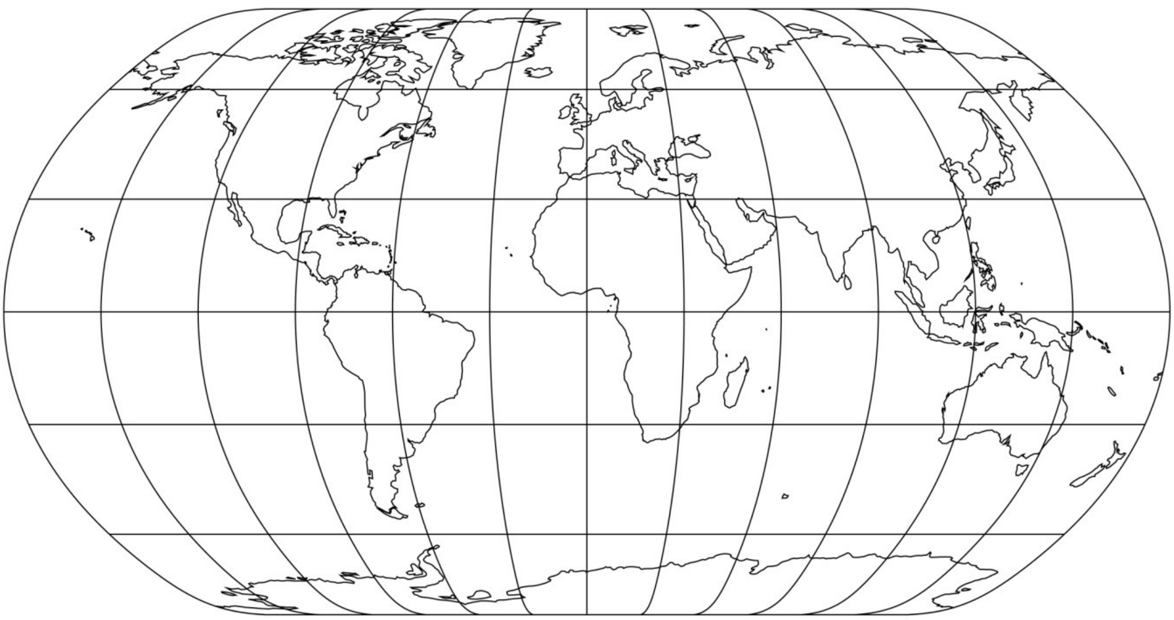


Figure 10: The graticule with coastlines computed with the polynomial Natural Earth projection.

5.2 Inverse projection

The inverse projection of the polynomial Natural Earth projection consists of the following steps:

(1) The initial guess for the unknown latitude: $\varphi_0 = Y \cdot R^{-1}$

(2) With the Newton-Raphson root finding algorithm, an improved latitude φ is calculated: $\varphi_{n+1} = \varphi_n - F(\varphi_n) \cdot (F'(\varphi_n))^{-1}$, where $F(\varphi_n)$ is the function from Equation 24, $F'(\varphi_n)$ its derivative, and $n = 0,1,2,\ldots,m$. At step m the iteration stops if $|\varphi_{m+1} - \varphi_m| < \varepsilon$, where ε is a sufficiently small positive quantity, typically close to the maximum precision of floating point arithmetic.

(3) The final latitude: $\varphi = \varphi_{m+1}$

(4) The final longitude: $\lambda = X \cdot R^{-1} \cdot (A_1 + A_2\varphi^2 + A_3\varphi^4 + A_4\varphi^{10} + A_5\varphi^{12})^{-1}$

The Newton-Raphson method is only applied to compute the latitude φ in step (2); the longitude λ can be computed in step 4 by inverting Equation 26. The Newton-Raphson method converges quickly with Equation 24. On average, less than four iterations are needed when transforming a regularly spaced graticule with 15 degrees resolution covering the whole sphere (with $\varepsilon = 10^{-13}$).

5.3 Comparison of the original and the polynomial Natural Earth projection

Since LSA is used for computing a polynomial equation, the two graticules are not identical. There are some deviations at corners where the lateral meridians meet the pole lines, which is due to the additional smoothness which was added to this projection.

Table 4: Comparison of the lengths and distances of parallels.

Latitude	The Lengths of Parallels			The Distances of Parallels from Equator		
	Original	New values	Difference	Original	New values	Difference
0	1	1	0.0000	0	0	0.0000
5	0.9988	0.9988	0.0000	0.062	0.0618	-0.0002
10	0.9953	0.9954	0.0001	0.124	0.1236	-0.0004
15	0.9894	0.9895	0.0001	0.186	0.1856	-0.0004
20	0.9811	0.9813	0.0002	0.248	0.2476	-0.0004
25	0.9703	0.9706	0.0003	0.31	0.3098	-0.0002
30	0.957	0.9573	0.0003	0.372	0.372	0.0000
35	0.9409	0.9413	0.0004	0.434	0.4342	0.0002
40	0.9222	0.9225	0.0003	0.4958	0.4962	0.0004
45	0.9006	0.9008	0.0002	0.5571	0.5575	0.0004
50	0.8763	0.8762	-0.0001	0.6176	0.618	0.0004
55	0.8492	0.8488	-0.0004	0.6769	0.677	0.0001
60	0.8196	0.8189	-0.0007	0.7346	0.7344	-0.0002
65	0.7874	0.7868	-0.0006	0.7903	0.7897	-0.0006
70	0.7525	0.7528	0.0003	0.8435	0.8429	-0.0006
75	0.716	0.7167	0.0007	0.8936	0.8934	-0.0002
80	0.6754	0.6763	0.0009	0.9394	0.94	0.0006
85	0.627	0.6256	-0.0014	0.9761	0.9786	0.0025
90	0.563	0.5504	-0.0126	1	1	0.0000

Table 4 presents the differences between the original tabular parameters and the parameters calculated from the polynomial coefficients in Table 3 by using the Equations 1, 26 and 27. At 0 degree of the length and at 90 degree of the distance, the results of additional constraints can be seen. Both values are unchanged, as expected. The value of the distance at zero degree is zero for the original and the approximated curves. This characteristic did not require an additional constraint in LSA. Since the polynomial does not contain a constant term in Equation 27, and all terms are multiplied by the latitude, at 0 degree the value of the parameter is zero. At higher latitudes (85 and 90 degrees) there are the largest differences, mainly caused by the improvement of the smoothness of the rounded corners. The biggest deviation of the length at 90 degrees is mainly caused by the manual reduction of the length of the pole line.

These conclusions can be made referring to Table 4. An objective evaluation of the polinomial graticule is difficult because the original tabular parameters do not have any units, and the graphical implications of the deviations are difficult to estimate. The evaluation was made by applying the graticule at a scale of 1 : 5,000,000, which shows the map of 3.489 m in width and 1.814 m in height. At this scale a difference in the length of parallels of 0.0001 is reflected along the border-meridian ($\lambda = 180°$) by a deviation of 0.35 mm in the X coordinate. Horizontal deviations are linearly decreasing to zero towards the central meridian ($\lambda = 0°$). For the distance of parallels, a difference of 0.0001 is equivalent to 0.18 mm of deviation along the Y coordinate.

The original Natural Earth projection was defined by 19 control points, which were used for the approximation. Figure 11 presents the absolute deviation from these points in the range of all positive latitudes. The graph shows large deviations at higher latitude, especially for the latitude at 90 degrees, which is almost 45 mm, caused by the described improvement of the rounded corners. Otherwise, the deviations for this graticule are less than 2.5 mm, mainly around the value of 1 mm at scale 1 : 5,000,000. At 60 and 65 degrees deviations are larger but they do not exceed the value of 2.5 mm.

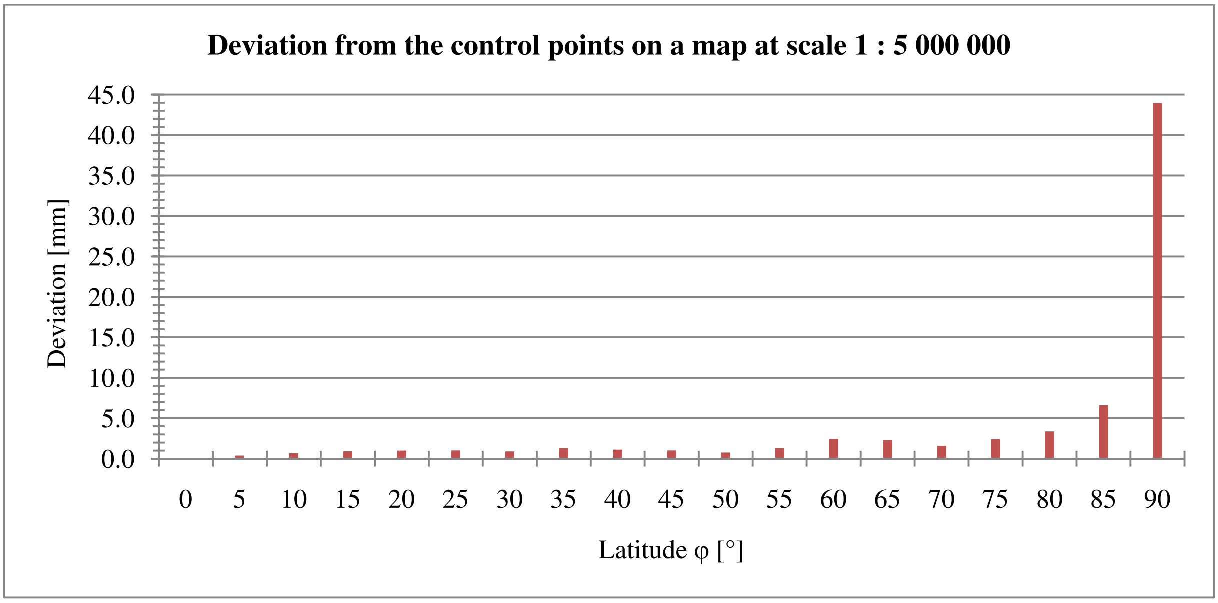


Figure 11: Deviation of the control points on the map at the scale 1: 5,000,000.

To evaluate whether the distortion parameter of the polynomial equation differs from the original projection, the new polynomial expressions were implemented in Flex Projector, and the distortion parameters were calculated. Java Code for implementing the polynomial equation in Flex Projector is presented in Appendix C. Table 5 provides the comparison of the distortion values from the original and new (improved) projections. As one can notice, the distortion values do not change significantly.

Table 5: Distortions for the Natural Earth projection computed with Flex Projector.

	Original projection	New projection
Weighted mean error for overall scale distortion	0.25	0.25
Weighted mean error for areal distortion	0.19	0.19
Mean angular deformation index	20.56	20.54

A graphical comparison of both projections in Flex Projector shows the close resemblance of both projection graticules. The corner where the pole line meets the meridians can be seen in the magnified original Natural Earth projection in Figure 12 (A). Picture B shows the smoothed corner of the polynomial projection. Figure 12 (C) illustrates the reduced pole line. Figure 12 (C) reveals that the largest deviations are at higher latitudes. The corner is smooth, but a small deviation from the original curve is present between 80 and 85 degrees.

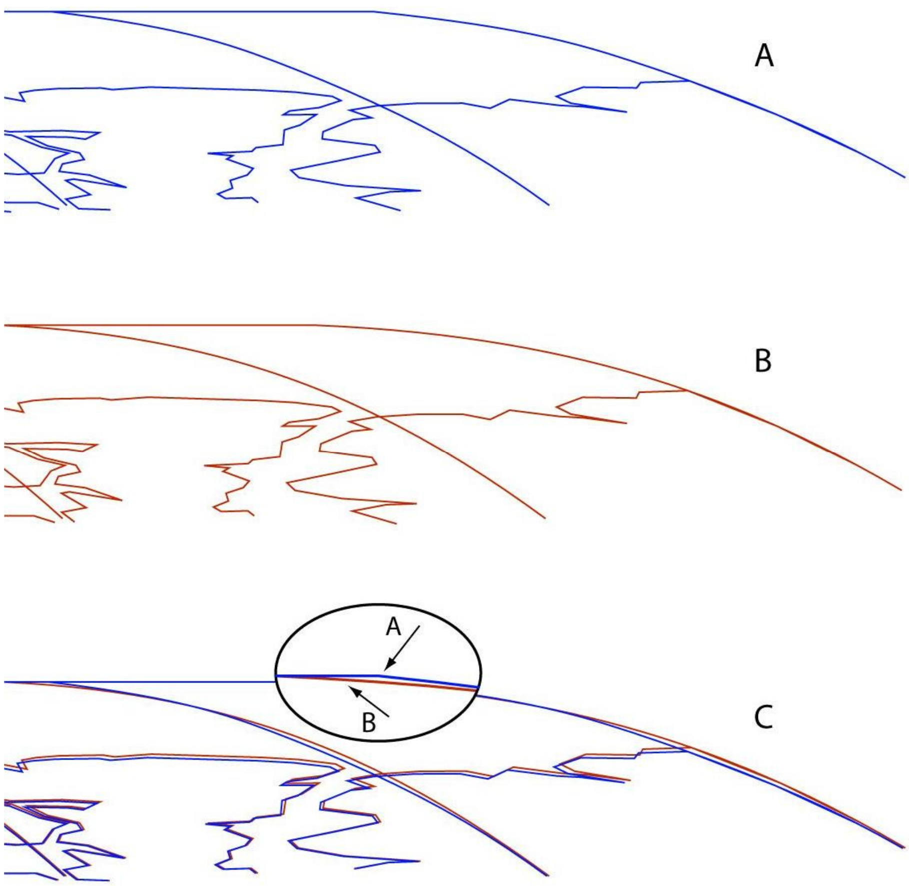


Figure 12: The original (A) and polynomial (B) Natural Earth projections are overlaid in (C). Arrows indicate changes in smoothness at the end of the pole line, which is shortened in (B).

6 CONCLUSION

This thesis had two goals. The first goal was to develop an equation for the Natural Earth projection with an inverse that it is easy to compute and implement in cartographic software. To achieve this goal, a polynomial expression was chosen to approximate this relatively new projection. Three basic characteristics were taken into account: the symmetry of the projection, the straight parallels, and the equally spaced meridians. According to those characteristics, Equations 26 and 27 were derived. For a good approximation of the original graticule, the order of the polynomials equals 13 and 11. The polynomial coefficients were defined by the LSA approximation method with additional constraints to preserve the width and height of the projection graticule. The reference variance and residuals from LSA were used to evaluate the differences between the approximated and the original projection. The equations are easy to code and fast to compute as only seven multiplications are required for each polynomial if factorized appropriately. The proposed inverse algorithm uses the Newton-Raphson method and requires only four iteration steps. It is slightly slower in comparison to an analytical inverse formula, but it converges quickly. On average, less than four iteration steps are required.

The Natural Earth projection expressed by the polynomial Equations 26 and 27 slightly deviates from Patterson's original projection by adding additional curvature to meridians where they meet the horizontal pole line. The curved corners are smoother than in the original design, which improves the visual appearance of the graticule. The polynomial equations were developed in collaboration with Tom Patterson and during this development he expressed his desire to smooth the edges at the ends of the pole lines. This second goal of this thesis was reached by changing the polynomial coefficients. In the functional model used for LSA, an additional constraint was added – fixing the slope of the Y coordinate at 7 degrees. Considering this constraint and the manual reduction of the pole line, the slope of meridians at the pole line becomes so small that the edges are no longer visible.

Since the scale distortion index, the areal distortion index, as well as the mean angular deformation index (Canters and Decleir, 1989) of the polynomial approximation of the Natural Earth projection are almost identical to those of the original projection, the authors of the paper "A polynomial equation of the Natural Earth projection" (Šavrič et al., accepted for publication) recommend using the polynomial equation (Equations 26 and 27 in Table 3) instead of the original Natural Earth projection. Details on LSA with constraints for obtaining the polynomial formulas are also provided in this paper to allow others to apply this technique to similar projections defined by tabular parameters. It remains to be explored how the polynomial approximation method can be generalized for any projection designed with Flex Projector.

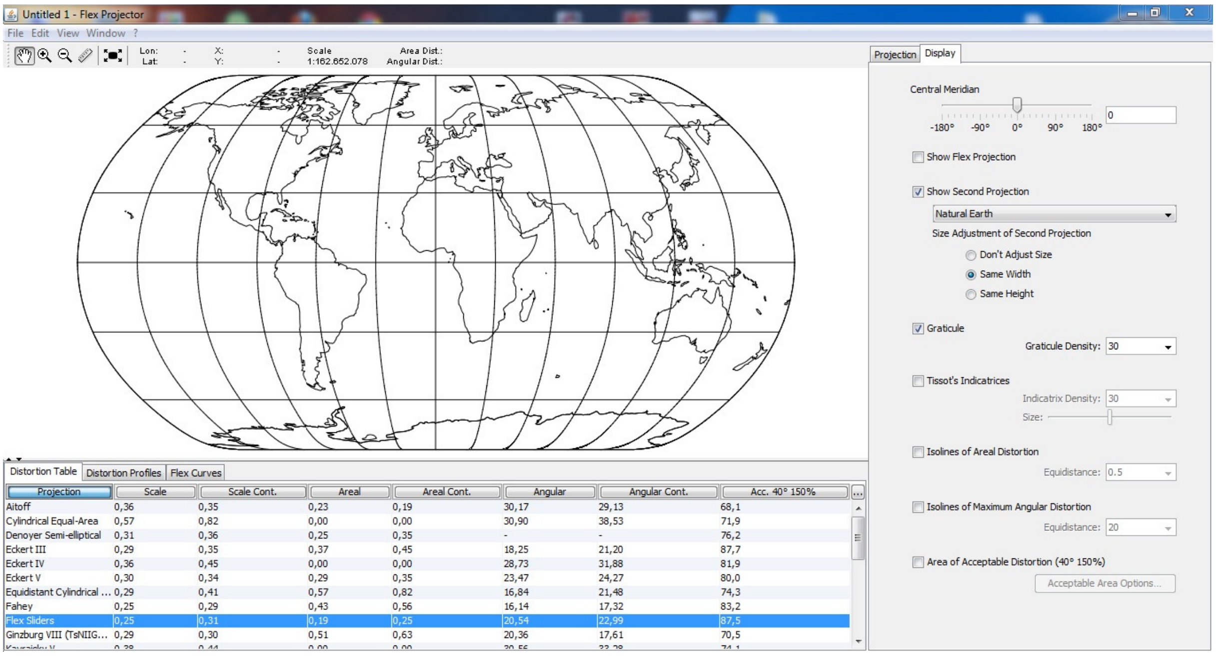


Figure 13: Screenshot of the Flex Projector with the improved Natural Earth projection.

The improved Natural Earth projection is implemented in the last version of the software application Flex Projector 0.118, released on 6 April 2011 (Jenny, B., Patterson, T. 2007). Figure 13 shows the screenshot of the Flex Projector with the Natural Earth projection.

Due to the fact that the Natural Earth projection now has its own polynomial equation, one can hope that the publication of the formulas will help the projection find its place in other cartographic projection libraries and software applications. Global Mapper is the first commercial software to include the developed polynomial Natural Earth projection. Developers of GeoCart, Natural Scene Designer, Java Map Projection Library (Appendix C), PROJ.4, MAPublisher, Geographic Imager and others have added the new Natural Earth projection to their software, which will be released in future updates. Figure 14 shows a screenshot of Global Mapper version 12.02.

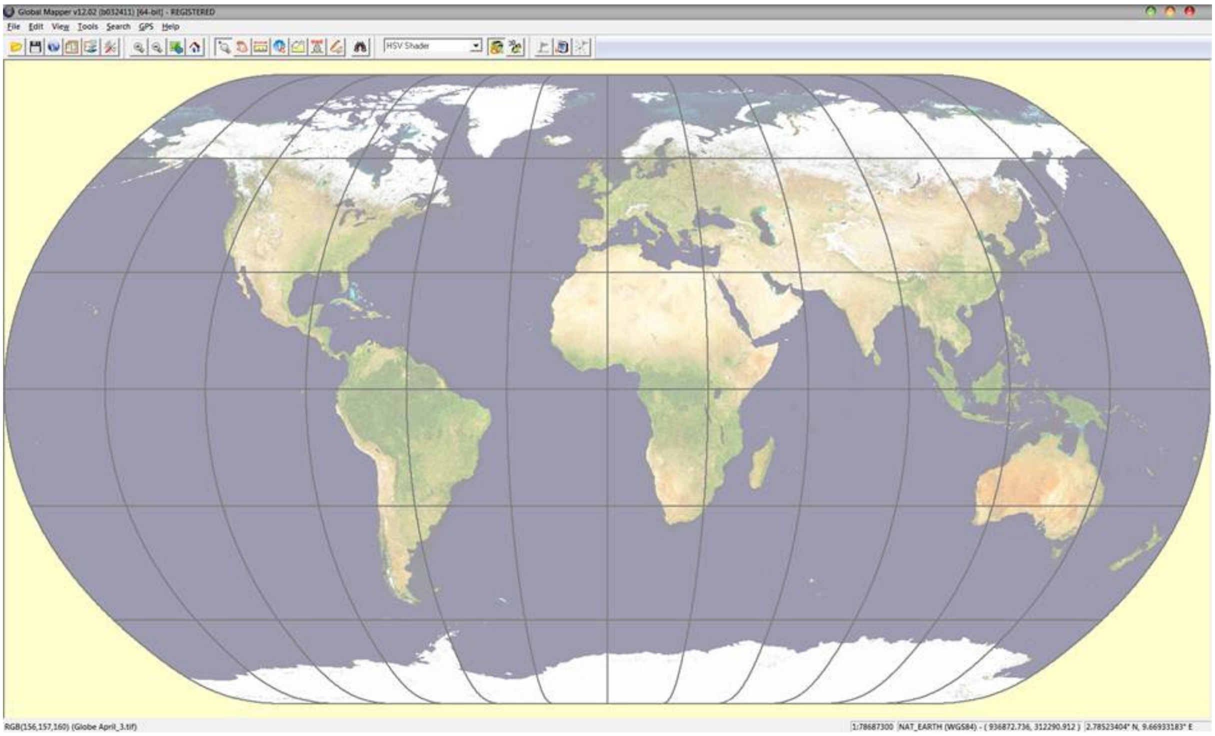


Figure 14: Screenshot of Global Mapper with the polynomial Natural Earth projection (Global Mapper, 2009).

7 RAZŠIRJEN POVZETEK DIPLOMSKE NALOGE V SLOVENSKEM JEZIKU

P.1 UVOD

P.1.1 Opis problema

V zgodovini so se kartografi pogosto soočali s težavami pri oblikovanju novih kartografskih projekcij. Pomanjkanje matematičnega znanja in zamudno prikazovanje rezultatov sta bila glavni težavi razvoja. Leta 1974 je Arthur H. Robinson predstavil nov način določanja kartografske projekcije, ki ga je uporabil za razvoj svoje, danes svetovno znane in uporabljane Robinsonove projekcije. Njegov pristop je povsem grafičen in čeprav je s tega vidika enostavnejši, ne omogoča eksaktne določitve analitične enačbe, ki povezuje sferne in projekcijske koordinate.

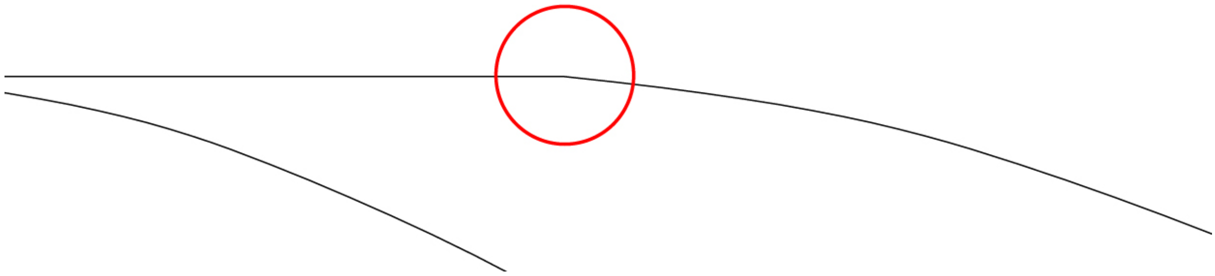
Tom Patterson je leta 2007 zasnoval Naravno Zemljino projekcijo z uporabo programa Flex Projector (Jenny et al., 2008). Ker program uporablja grafični pristop Robinsona, Naravna Zemljina projekcija ni določena z analitično enačbo. Zasnovana je na podlagi dveh parametrov – dolžine paralel in njihove oddaljenosti od ekvatorja. Oba parametra sta podana za vsakih pet stopinj geografske širine in kot taka definirata pare kontrolnih točk. Krivulje meridianov skozi te točke določa interpolacijska metoda kubičnih zlepkov. Ta metoda interpolacije je enostavna za preračun, vendar vsebuje veliko število parametrov in zahteva kar precej programiranja, če želimo projekcijo uporabiti v drugih programskih orodjih. Tako uporaba te projekcije ostaja omejena na program, v katerem je bila zasnovana.

P.1.2 Glavni cilj naloge

Z namenom, da bi razširili uporabo Naravne Zemljine projekcije tudi na ostale programske pakete, je glavni cilj te diplomske naloge določitev analitične enačbe projekcije. Analitična enačba mora omogočati transformacijo sferičnih v katarzične koordinate za celotno definicijsko območje. Pretvorba mora biti enostavna, s čim manj parametri in imeti določeno tudi inverzno funkcijo. Programiranje naj bo enostavno in hitro.

P.1.3 Dodatni cilj naloge

Posebnost Naravne Zemljine projekcije je stičišče robnega meridiana in pola, ki je v tej projekciji predstavljen kot daljica. Ob večji povečavi se na tem stiku pojavi nezaželeni lom (Slika P. 1). Snovalec projekcije v programu Flex Projector loma ni uspel zgladiti, zato je dodatni cilj te naloge izboljšati gladkost omenjenega stika.



Slika P. 1: Povečava stičišča projekcij meridiana in pola. Opazen lom je označen z rdečim krogom.

P.2 ZASNOVA IN IZVOR NARAVNE ZEMLJINE PROJEKCIJE

P.2.1 Naravna Zemljina kartografska projekcija

Naravno Zemljino projekcijo je zasnoval Tom Patterson leta 2007 kot poskus izdelave projekcije, ki bi bolje kot vse dosedanje zadovoljila pogoje za prikaz karte celotnega fizičnega površja Zemlje v majhnem merilu (Jenny et al., 2008). Za sam nastanek projekcije je bilo uporabljeno iterativno orodje za grafično oblikovanje kartografskih projekcij, imenovano Flex Projector. Projekcija je kombinacija dveh že obstoječih kartografskih projekcij – Kavraiskiy VII in Robinsonove (Jenny et al., 2008).

Obliko mreže meridianov in paralel določata dva parametra: (1) dolžina paralel (l_φ) in (2) oddaljenost paralel od ekvatorja (d_φ). Vrednosti obeh parametrov so za Naravno Zemljino projekcijo podane na vsakih pet stopinj geografske širine in so brez enot (Tabela P. 1). Parametri so bili določeni z iterativnimi poizkusi in na podlagi vrednotenja podob kontinentov na sami projekciji (Jenny et al. 2008). Njihovo pretvorbo v projekcijske koordinate podajata spodnji enačbi:

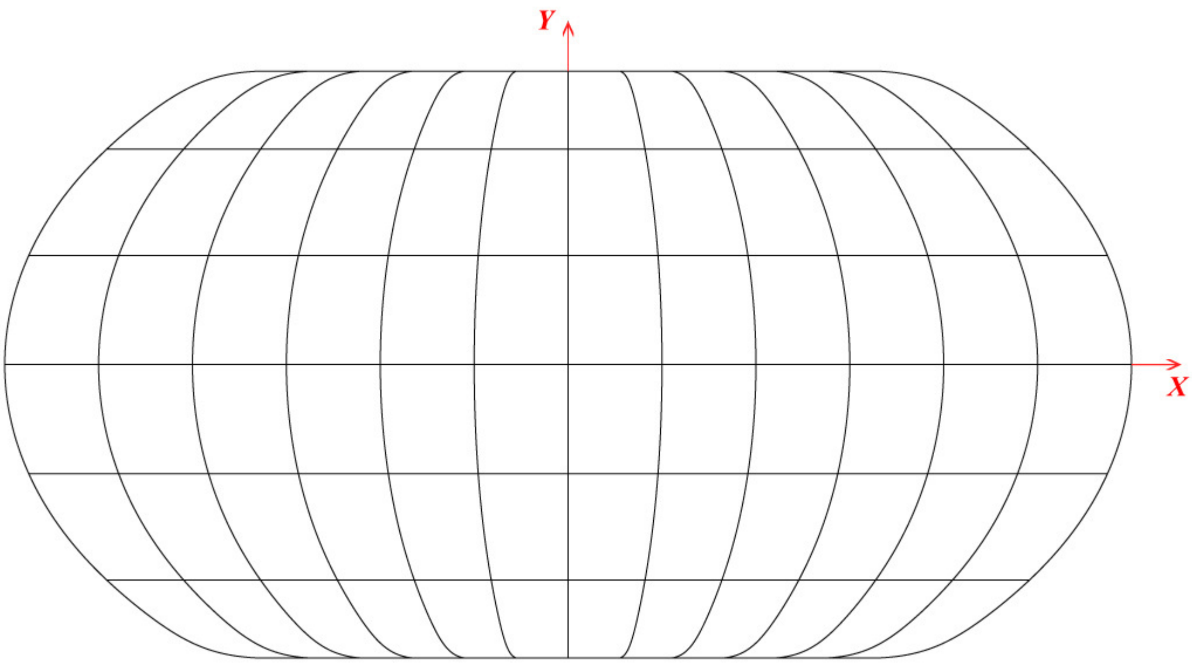
$$X = R \cdot s \cdot l_\varphi \cdot \lambda, \qquad \text{(Enačba P. 1)}$$
$$Y = R \cdot s \cdot d_\varphi \cdot k \cdot \pi.$$

Tabela P. 1: Vrednost parametrov za Naravno Zemljino projekcijo (Jenny, et al., 2008, stran 25).

Geografska širina	Dolžina paralel	Oddaljenost paralel od ekvatorja
0	1	0
5	0.9988	0.062
10	0.9953	0.124
15	0.9894	0.186
20	0.9811	0.248
25	0.9703	0.31
30	0.957	0.372
35	0.9409	0.434
40	0.9222	0.4958
45	0.9006	0.5571
50	0.8763	0.6176
55	0.8492	0.6769
60	0.8196	0.7346
65	0.7874	0.7903
70	0.7525	0.8435
75	0.716	0.8936
80	0.6754	0.9394
85	0.627	0.9761
90	0.563	1

V Enačbi P. 1 sta X in Y projekcijski ter φ in λ sferni koordinati. R je polmer Zemljine krogle, medtem ko $s = 0{,}8707$ predstavlja linijsko merilo na ekvatorju in določa lego sekantnega valja z dvema standardnima paralelama, $k = 0{,}52$ pa je razmerje med dolžino srednjega meridiana in ekvatorja.

Oba parametra l_φ in d_φ podajata pare kontrolnih točk skozi katere potekajo krivulje meridianov. Vmesne točke so pri osnovni definiciji podane z interpolacijsko metodo kubičnih zlepkov, ki jo za ta namen uporablja tudi Flex Projector. Tabela P. 1 in Enačba P. 1 skupaj z interpolacijsko metodo tako predstavljata transformacijo med sferičnimi in projekcijskimi koordinatami, ki jo želimo v tej diplomski nalogi poenostaviti.



Slika P. 2: Mreža meridianov in paralel Naravne Zemljine projekcije za vsakih 30° geografske širine in dolžine. Z rdečo barvo sta prikazani tudi koordinatni osi.

Kot je razvidno iz zgornje slike, je Naravna Zemljina projekcija psevdocilindrična projekcija z enakomerno razporeditvijo meridianov ter ravnimi paralelami. Osi X in Y predstavljata projekcijo ekvatorja in srednjega meridiana (Slika P. 2). Mreža je simetrična glede na obe koordinatni osi. Dolžina ekvatorja je zmanjšana za merilo s, medtem ko se srednji meridian preslika kot ravna premica, in je 0,52-kratnik dolžine ekvatorja. Ostali meridiani so s svojo konkavno stranjo obrnjeni proti srednjemu meridianu. Stičišča robnih meridianov in daljic polov so zakrivljena, kar je prepoznavna lastnost te projekcije. Zakrivljenost ostalih meridianov se zmanjšuje takrat, ko se od roba približujemo srednjemu meridianu, kateri pravokotno seka daljici polov.

Naravna Zemljina projekcija ni niti konformna niti ekvivalentna. Njene distorzijske vrednosti so primerljive z ostalimi projekcijami. Kot pri vseh podobnih psevdocilindričnih projekcijah, so največje deformacije prisotne na polih (Jenny et al., 2008). Slike 3, 4 in 5 na straneh 7 in 8 v osnovnem delu te diplomske naloge prikazujejo razporeditev vrednosti deformacij.

P.2.2 Analitične enačbe za Robinsonovo projekcijo

Naravna Zemljina projekcija zagotovo ni prva takšna projekcija, ki nima osnovne analitične enačbe. Prvo takšno projekcijo je ustvaril sam snovalec grafičnega pristopa, Arthur H. Robinson. Njegova Robinsonova projekcija je bila že od vsega začetka predmet debat, v katerih so se spraševali kako uspešno modelirati projekcijo skozi pare kontrolnih točk. V svojem bistvu se ločita dva načina modeliranja – aproksimacija in interpolacija. Interpolacijska metoda kubičnih zlepkov je na primeru Naravne Zemljine projekcije že uporabljena in tudi neprimerna zaradi velikega števila parametrov ter oteženega programiranja. Veliko enostavnejše so aproksimacijske metode, pa čeprav te ne podajo krivulje, ki bi potekale natančno skozi kontrolne točke.

Za Robinsonovo projekcijo sta bili razviti dve aproksimaciji. Prvo sta predstavila Canters in Decleir (1989). Gre za aproksimacijo z dvema polinomskima enačbama pete stopnje, ki skupno vsebujeta le šest parametrov. Drugo rešitev aproksimacije je dve leti za tem predstavil Beineke (1991, 1995). Za X koordinato je uporabil polinomsko enačbo sedme stopnje, medtem ko je Y koordinato izrazil z eksponentno enačbo. Čeprav je druga rešitev nekoliko boljša, je izračun prve enostavnejši in hitrejši. To dokazuje tudi manjša simulacija v Java programskem jeziku. Ob enakem številu parametrov je eksponentna enačba nekaj več kot desetkrat počasnejša za preračun od polinomske. Edina slabost polinomskih enačb je ta, da zanje pogosto ne obstaja inverzna funkcija. Ta težava se v večini primerov rešuje z uporabo drugih numeričnih metod reševanja, kot sta recimo tangentna (Newtonova) metoda in metoda bisekcije. Glede na vsa zgoraj navedena dejstva smo se odločili za izbiro polinomske oblike enačbe za določitev Naravne Zemljine projekcije.

P.3 UPORABLJENE NUMERIČNE METODE

P.3.1 Metoda najmanjših kvadratov – posredna izravnava

V prvih fazah razvoja enačb je bila uporabljena posredna izravnava po metodi najmanjših kvadratov. Število enačb v tem matematičnem modelu je enako številu podanih vrednosti za vsak parameter (n). Vrednost parametra se obravnava kakor opazovanje, polinomski koeficienti pa nastopajo kot neznanke (u). Enačba P. 2 predstavlja matrični zapis, ki se ga rešuje po Enačbi P. 3 (Mikhail, Ackerman, 1976).

$$\boldsymbol{v} + \boldsymbol{B} \cdot \boldsymbol{\Delta} = (-\boldsymbol{l} + \boldsymbol{d}) = \boldsymbol{f}, \tag{Enačba P. 2}$$

$$\boldsymbol{\Delta} = \boldsymbol{N}^{-1} \cdot \boldsymbol{t} = \left(\boldsymbol{B}^{\boldsymbol{T}} \cdot \boldsymbol{P} \cdot \boldsymbol{B}\right)^{-1} \cdot \boldsymbol{B}^{\boldsymbol{T}} \cdot \boldsymbol{P} \cdot \boldsymbol{l}, \tag{Enačba P. 3}$$
$$\boldsymbol{v} = \boldsymbol{B} \cdot \boldsymbol{\Delta} - \boldsymbol{f},$$

Vektor $\boldsymbol{l}$ dimenzije $n \times 1$ je vektor opazovanj, $\boldsymbol{\Delta}$ dimenzije $u \times 1$ je vektor u neznank, $\boldsymbol{B}$ je matrika koeficientov $n \times u$ z rangom u, vektor $\boldsymbol{f}$ predstavlja $n \times 1$ vektor odstopanj in $\boldsymbol{d}$ je vektor konstant, ki nastopajo v enačbah opazovanj. Pri izračunu si pomagamo z matriko koeficientov normalnih enačb $\boldsymbol{N}$ dimenzije $u \times u$ ter z vektorjem stalnih členov $\boldsymbol{t}$ ($u \times 1$). Matrika $\boldsymbol{P}$ je matrika uteži med opazovanji (Mikhail, Ackerman, 1976).

Ker so vse vrednosti parametrov izražene s polinomi, gre v tem konkretnem primeru za linearni sistem enačb, ki za rešitev ne potrebuje približnih vrednosti neznank in iterativnega procesa reševanja. Matrika uteži $\boldsymbol{P}$ je enaka enotski matriki, saj so posamezne vrednosti parametrov neodvisne druga od druge in določene na enak način. Za medsebojno primerjavo rezultatov aproksimacij po tej metodi uporabljamo vrednost največjega popravka iz vektorja $\boldsymbol{v}$ in pa referenčno varianco, določeno po Enačbi P. 4, kjer r predstavlja nadštevilnost.

$$\hat{\sigma}_0^2 = \frac{\boldsymbol{v}^T \cdot \boldsymbol{P} \cdot \boldsymbol{v}}{r} \tag{Enačba P. 4}$$

P.3.2 Posredna izravnava s funkcijsko odvisnimi neznankami

Ta, nekoliko dograjen model posredne izravnave, je bil uporabljen za končno določitev vrednosti polinomskih koeficientov. Enačbam opazovanj smo dodali nov sistem t.i. veznih enačb (n'), ki opisuje medsebojno odvisnost neznank (Enačba P. 5). Matrika $\boldsymbol{C}$ je matrika koeficientov, $\boldsymbol{g}$ pa vektor konstant. S tem povečamo tudi nadštevilnost enačb, saj velja povezava: $n + n' = r + u$. Ob tem zagotovimo izbrane medsebojne pogoje, ki jih želimo doseči ali pa aproksimacijsko krivuljo položimo natančno skozi izbrano točko. Oba sistema enačb (Enačbi P. 2 in 5) rešimo po Enačbi P. 6 (Mikhail, Ackerman, 1976).

$$\boldsymbol{C} \cdot \boldsymbol{\Delta} = \boldsymbol{g} \tag{Enačba P. 5}$$

$$\begin{array}{lll} \boldsymbol{v} + \boldsymbol{B} \cdot \boldsymbol{\Delta} = \boldsymbol{f} & \boldsymbol{N} = \boldsymbol{B}^{\boldsymbol{T}} \cdot \boldsymbol{B} & \boldsymbol{\Delta}_{\boldsymbol{0}} = \boldsymbol{N}^{-\boldsymbol{1}} \cdot \boldsymbol{B}^{\boldsymbol{T}} \cdot \boldsymbol{l} \\ \boldsymbol{C} \cdot \boldsymbol{\Delta} = \boldsymbol{g} & \boldsymbol{M} = \boldsymbol{C} \cdot \boldsymbol{N}^{-1} \cdot \boldsymbol{C}^T & \boldsymbol{\delta\Delta} = \boldsymbol{N}^{-1} \cdot \boldsymbol{C}^T \cdot \boldsymbol{M}^{-1} \cdot (\boldsymbol{g} - \boldsymbol{C} \cdot \boldsymbol{\Delta}_0) \\ \boldsymbol{\Delta} = \boldsymbol{\Delta}_{\boldsymbol{0}} + \boldsymbol{\delta\Delta} & \boldsymbol{v} = \boldsymbol{B} \cdot \boldsymbol{\Delta} - \boldsymbol{f} & \end{array} \tag{Enačba P. 6}$$

P.3.3 Tangentna (Newtonova) metoda reševanja enačb

Tangentna metoda je numerična metoda reševanja enačb oblike $f(x) = 0$. Temelji na iterativnem načinu reševanja z le enim začetnim približkom rešitve. Pogosto se uporablja za reševanje polinomskih enačb višjih redov, kadar zanje ne obstaja inverzna funkcija. Če je približek dovolj blizu pravi rešitvi in je naklon krivulje na tem območju enakega predznaka, iteracija kvadratično konvergira proti končni rešitvi (Bronštejn et al. 1974). Ta metoda je v našem primeru uporabljena za inverzno transformacijo iz kartezičnih v sferne koordinate. Opisuje jo Enačba P. 7, kjer je $F(x_{n-1})$ podana enačba in $F'(x_{n-1})$ njen prvi odvod. x_{n-1} in x_n predstavljata začetno in novo rešitev enačbe, n in $(n-1)$ pa sta iteracijska koraka.

$$x_n = x_{n-1} - F(x_{n-1}) \cdot \left(F'(x_{n-1})\right)^{-1} \rightarrow F(x_{n-1}) = 0 \,, \qquad \text{(Enačba P. 7)}$$

V geometrijskem pomenu, metoda ne išče presečišča funkcije z abscisno osjo, temveč presečišče tangente na krivuljo v začetni točki s to osjo. Od tu tudi izhaja ime metode (Bronštejn et al. 1974).

P.4 APROKSIMACIJA NARAVNE ZEMLJINE PROJEKCIJE S POLINOMSKO ENAČBO

Pri določitvi analitične enačbe za Naravno Zemljino projekcijo je bila uporabljena aproksimacija s polinomsko enačbo. Pri tem smo upoštevali dva kriterija, in sicer kot prvi majhno število členov v enačbi, kot drugi pa absolutno razliko med originalno in nadomestno polinomsko določeno projekcijo, ki naj bo čim manjša. Prvi kriterij zagotavlja poenostavitev in hitrejši izračun, drugi pa rezultat, ki odraža podobnost.

Pri razvoju enačbe je sodeloval snovalec projekcije, Tom Patterson. Ker je cilj naloge tudi izboljšava zakrivljenih robov, ima pri nalogi njegova grafična ocena veliko večji pomen kot absolutna razlika med originalno in nadomestno, s polinomsko enačbo določeno projekcijo (v nadaljevanju uporabljamo izraz polinomska Naravna Zemljina projekcija). Celoten proces določitve vsebuje šest ločenih faz, ki so predstavljene v naslednjih podpoglavjih.

P.4.1 Oblikovanje polinomskih enačb

Obe projekcijski koordinati X in Y sta izraženi z dvema sfernima, φ in λ. Tako splošna polinomska enačba (Enačba P. 8) za primer tretje stopnje vsebuje 20 členov za obe projekcijski koordinati. Vendar pa moramo splošno obliko nekoliko preoblikovati, saj so zaradi lastnosti psevdocilindrične projekcije vrednosti nekateri polinomskih koeficientov blizu ničle in je njihova uporaba tako nesmiselna. Pri oblikovanju obeh polinomskih enačb smo zato upoštevali naslednje tri lastnosti Naravne Zemljine projekcije (Canters, 2002):

(1) paralele so ravne, a ne enakomerno porazdeljene daljice,
(2) meridiani so enakomerno porazdeljeni in
(3) projekcija je simetrična glede na obe koordinatni osi.

$$p(\varphi, \lambda) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} c_{ij} \lambda^i \varphi^j \qquad \text{(Enačba P. 8)}$$

Iz prve lastnosti izhaja, da je Y koordinata odvisna le od geografske širine φ, zato iz nje odstranimo vse člene, ki vsebujejo geografsko dolžino. Druga lastnost spreminja X koordinato. Zato, ker so meridiani enakomerno porazdeljeni, ta enačba zato vsebuje le linearne člene geografske dolžine. Na koncu pa imamo še simetrijo. Zaradi simetrije skozi obe osi, koordinata X vsebuje le člene s sodo, koordinata Y pa z liho potenco geografske širine (Canters, 2002). Rezultat za tretjo stopnjo polinoma prikazuje Enačba P. 9.

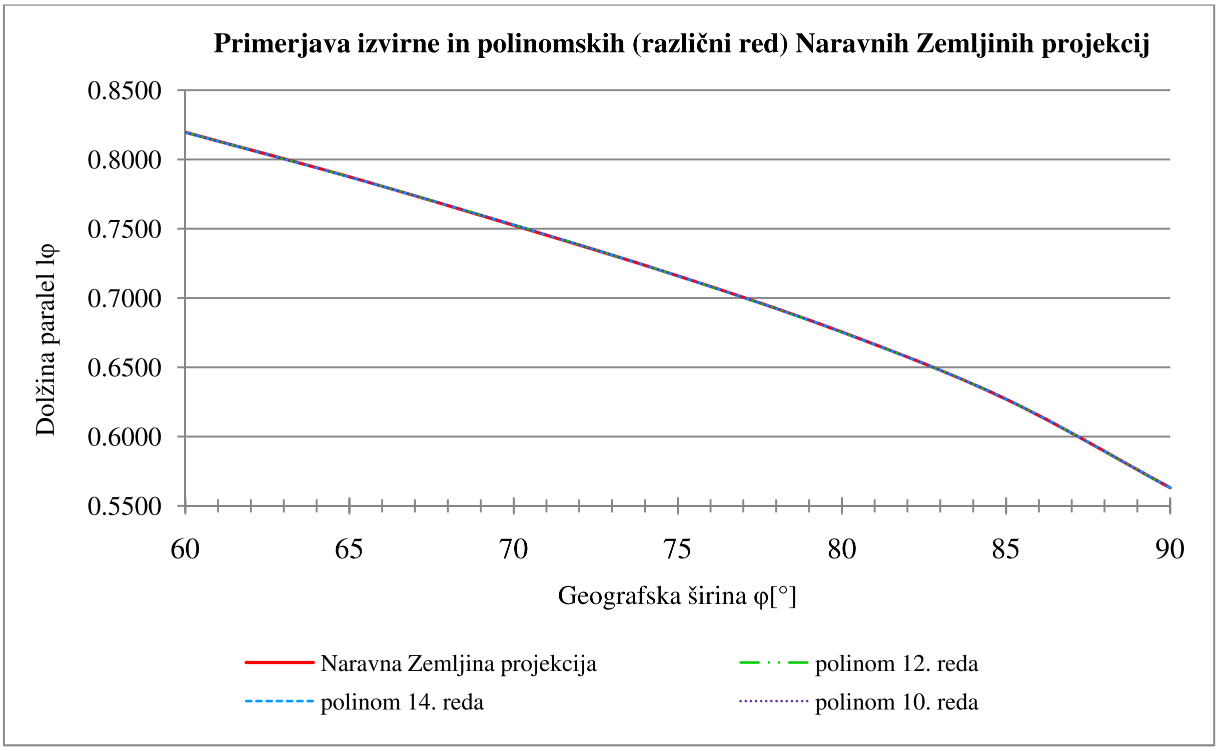
$$\begin{aligned} X &= \lambda \cdot (A_{10} + A_{12}\varphi^2 + \cdots) \\ Y &= B_{01}\varphi + B_{03}\varphi^3 + \cdots \end{aligned} \qquad \text{(Enačba P. 9)}$$

P.4.2 Povečanje polinomske stopnje

Tretja stopnja polinoma nikakor ne zadošča za uspešno aproksimacijo, saj želimo zmanjšati tudi absolutno razliko med projekcijsko mrežo izvorne in polinomske Naravne Zemljine projekcije. Pri nižjih stopnjah polinoma je aproksimacijska krivulja precej različna od izvorne. Po drugi strani pa prevelika stopnja prinese krivuljo, ki valovi med kontrolnimi točkami, kar pa tudi ni želen rezultat. S postopnim dvigovanjem stopnje polinoma smo tako iskali tisti polinom, pri katerem so razlike še relativno majhne in je krivulja najbolj podobna originalni.

Za vsako koordinato posebej smo ločeno izvedli posredno izravnavo z različnimi stopnjami polinomov. Za primerjavo rezultatov so bile uporabljene vrednosti največjih popravkov posamezne izravnave in referenčna varianca. Ko se z večjo stopnjo polinoma vrednosti teh količin niso več bistveno spremenile, so bile enačbe vpeljane v program Flex Projector. Ta je omogočil grafični izris koordinatne mreže in s tem vizualno primerjavo izvorne in polinomske koordinatne mreže. Tabela P. 2 in Slika 8 na strani 20 prikazujeta primerjavo različnih stopenj. Na Sliki P. 3 vidimo grafično primerjavo zadnjih treh redov polinoma, kjer grafične razlike med različnimi redi niso več opazne.

Za Naravno Zemljino projekcijo je bila po grafični oceni izbrana dvanajsta stopnja polinoma (samo pri geografski širini φ).



Slika P. 3: Primerjava izvirne in polinomskih Naravnih Zemljinih projekcij. Pri uporabljenih polinomih 10., 12., in 14. stopnje razlike niso več opazne.

Tabela P. 2: Referenčne variance in največji popravki za različne polinomske stopnje po izravnavi. Zadnja vrstica prikazuje nove vrednosti po odstranitvi členov z majhnim prispevkom za 12. stopnjo. Opomba: Enote so mm na karti v merilu 1 : 5.000.000, kjer je polmer Zemlje enak 6.378.137 metrov.

Polinomska stopnja	Dolžina paralel (l_φ)		Oddaljenost paralel (d_φ)	
	$\hat{\sigma}_0^2$ [mm]	max (v) [mm]	$\hat{\sigma}_0^2$ [mm]	max (v) [mm]
6.	7,322	14,701	2,564	5,483
8.	2,328	4,556	1,323	2,447
10.	0,494	1,186	0,499	0,760
12.	0,406	1,173	0,291	0,813
14.	0,359	0,911	0,186	0,476
*12.	0,406	1,081	0,287	0,823

P.4.3 Odstranitev vmesnih členov in končna oblika enačb

Z dvigovanjem polinomske stopnje zmanjšamo absolutno razliko, a s tem povečamo število členov v enačbah. Nekateri vmesni členi prispevajo h končnemu rezultatu (vrednosti koordinate) manj kakor ostali. Z namenom poenostavitve izračuna in zmanjšanja števila členov v enačbi so bili v tej fazi odstranjeni posamezni vmesni polinomski členi.

Majhne vrednosti koeficientov po izravnavi s polinomom dvanajste stopnje nakazujejo možnost odstranitve šeste in osme potence geografske širine za X koordinato. Polinomu, ki izraža Y koordinato pa smo odstranili peto potenco. Po ponovni izravnavi z novimi polinomi se projekcijska mreža ni bistveno spremenila (Tabela P. 2, vrednosti za stopnjo *12.). Vrednosti referenčne variance za oba primera sta se celo izboljšala. Enačba P. 10 prikazuje končno obliko enačb, kjer oba polinoma nadomeščata parametra l_φ in d_φ ter konstante s, k in π.

$$X = R \cdot \lambda \cdot (A_1 + A_2\varphi^2 + A_3\varphi^4 + A_4\varphi^{10} + A_5\varphi^{12}) \qquad \text{(Enačba P. 10)}$$
$$Y = R \cdot (B_1\varphi + B_2\varphi^3 + B_3\varphi^7 + B_4\varphi^9 + B_5\varphi^{11})$$

P.4.4 Dodajanje pogojev v izravnavo

Uporaba aproksimacijske metode in posredne izravnave v nobenem primeru ne prinaša enačbe, ki bi podala popolnoma enake dimenzije in obliko projekcijske mreže. Na prav vsaki kontrolni točki se polinomska mreža razlikuje od originalne za vrednost popravka po izravnavi. Kljub vsemu želimo, da bi ohranili vsaj enake dimenzije projekcije in s tem vsaj delno zmanjšali razlike med projekcijama. Za dosego tega smo v posredno izravnavo vpeljali dva dodatna pogoja, torej, da se dolžini ekvatorja in srednjega meridiana morata ohraniti. Z drugimi besedami to pomeni, da se dolžina paralele pri 0° in oddaljenost paralele pri 90° geografske širine ohranjata.

S kombinacijo Enačb P. 1 in 10 oba pogoja zapišemo v obliki:

$$\varphi = 0 \rightarrow l_\varphi = 1 \rightarrow A_1 = s \qquad \text{(Enačba P. 11)}$$

$$\varphi = \frac{\pi}{2} \rightarrow d_\varphi = 1 \rightarrow$$
$$B_1 \cdot \frac{\pi}{2} + B_2\left(\frac{\pi}{2}\right)^3 + B_3\left(\frac{\pi}{2}\right)^7 + B_4\left(\frac{\pi}{2}\right)^9 + B_5\left(\frac{\pi}{2}\right)^{11} = \mathrm{s} \cdot \mathrm{k} \cdot \pi$$

Ker v izravnavi uporabljamo linearne enačbe, oba pogoja izrazimo kot medsebojno odvisnost parametrov. Od te faze naprej se vrednosti polinomskih koeficientov določajo po posredni metodi izravnave s funkcijsko odvisnimi neznankami (Enačba P. 6).

P.4.5 Zgladitev stičišča robnega meridiana in projekcije pola

Da bi se izboljšala gladkost stičišča robnega meridiana in pola, mora biti krivulja robnega meridiana $Y = f(X)$ oblikovana na način, da se v stičišču z daljico pola njena vrednost naklona ustavi natanko pri nič stopinjah. To pomeni, da mora biti odvod v tej točki enak nič. Žal pa ne poznamo funkcije Y v odvisnosti od X koordinate.

Ker imamo za vsako koordinato svojo parametrično funkcijo, moramo v tem primeru ugotoviti, katera je tista, ki ima največji vpliv na končno vrednost naklona. Tako razširimo parcialni odvod Y koordinate z verižnim pravilom, kakor je prikazano v Enačbi P. 12. Geografsko dolžino pri tem obravnavamo kot konstanto $\lambda = 180°$, saj se ta situacija rešuje le za primer robnega meridiana in je zato ne navajamo v nadaljnjih enačbah.

$$\frac{dY}{d\varphi} = \frac{dY}{dX} \cdot \frac{dX}{d\varphi} \qquad \text{(Enačba P. 12)}$$

Funkciji obeh koordinat sta medsebojno neodvisni, zato sprememba ene ne vpliva na drugo. Obe pa vplivata na funkcijo Y v odvisnosti od X koordinate. Podobno velja tudi za odvode, ki so predstavljeni v Enačbi P. 12. Iz te enačbe tako sledi, da sprememba odvoda Y koordinate po geografski širini $\frac{dY}{d\varphi}$ povzroči spremembo vrednosti odvoda $\frac{dY}{dX}$, v našem primeru je to vrednost naklona robnega meridiana. Pri tem odvod X koordinate po geografski širini $\frac{dX}{d\varphi}$ ostaja ves čas enak.

Popolno analitično gladkost robov dosežemo, če je naklon robnega meridiana v točki pola enak natančno nič. V izravnavo tako dodamo pogoj, ki odvod Y koordinate v tej točki nastavi na vrednost nič: $\frac{dy}{d\varphi} = 0$. Po Enačbi P. 12 tako spremenimo tudi naklon robnega meridiana. Vendar pa tak dodaten pogoj v izravnavi povzroči nezaželeno odstopanje (izboklino) na mreži projekcije. Ta razlika je precejšnja in posledica tega so spremenjena oblika projekcije in prevelika odstopanja v kontrolnih točkah. Ker vseeno želimo prikriti lom, smo se odločili za kompromisno rešitev. Naklon grafa smo najprej povečali na sedem stopinj. Zato, da bi prikrili nastalo izboklino, smo nato še dodatno zmanjšali dolžino daljice pola. Sedem stopinjski naklon omogoči, da uporabnik loma ne zazna pri veliki, a še smiselno uporabljeni povečavi. Rezultat je tako boljši kot pri izvorni projekciji, kjer je bilo lom mogoče dokaj hitro zaznati.

Za grafično gladkost zakrivljenih robov Naravne Zemljine projekcije je bilo tako potrebno dvoje. Najprej smo v izravnavo dodali tretji pogoj: naklon krivulje Y koordinate v odvisnosti od geografske širine na polu (v točki $\varphi = 90°$) je bil določen na 7 stopinj (Enačba P. 13). Ker pa vsiljen naklon povzroči manjšo izboklino na robnem meridianu je bila dolžina paralele na polu reducirana iz 0.563 na 0.550 že pred samo izravnavo. S tem se manjše odstopanje prikrije in izboklina je potisnjena navznoter. Rezultat teh dveh dodatnih sprememb je polinomska aproksimacija, ki obenem tudi izboljšuje mrežo Naravne Zemljine projekcije in je navedena v poglavju P.5.1 kot rezultat te diplomske naloge.

$$B_1 + B_2 \cdot \frac{3\pi^2}{4} + B_3 \cdot \frac{7\pi^6}{64} + B_4 \cdot \frac{9\pi^8}{256} + B_5 \cdot \frac{11\pi^{10}}{1024} = \tan(7°)$$ (Enačba P. 13)

P.4.6 Inverzna projekcija

Za določitev inverznih enačb projekcije izhajamo iz sistema dveh enačb (Enačba P. 10), kjer projekcijski koordinati X in Y nastopata kot dani vrednosti ter sferni koordinati φ in λ kot neznanki. Enačba Y koordinate vsebuje le eno neznanko, φ, katere vrednost nato uporabimo v drugi enačbi za izračun neznanke λ. Princip je v svoji osnovi enostaven, vendar zahteva invertiranje polinomske enačbe 11. stopnje.

Na splošno ne obstaja analitični inverzni način reševanje polinomskih enačb z realnimi koeficienti. Takšne sisteme se zato pogosto rešuje z različnimi numeričnimi metodami. Za invertiranje Naravne Zemljine projekcije smo izbrali tangentno metodo, saj metoda izpolnjuje pogoj konvergence. Odvod enačbe je na celotnem definicijskem območju $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ pozitiven in zato tukaj ni nobenega lokalnega ekstrema. Hkrati je kvocient $Y \cdot R^{-1}$ lahko uporabljen kot začetna vrednost, saj so vse njegove vrednosti znotraj definicijskega območja geografske širine.

P.5 ANALITIČNA ENAČBA NARAVNE ZEMLJINE PROJEKCIJE – REZULTATI

P.5.1 Polinomski enačbi projekcije

Enačba P. 14 nadomešča tabelarični parameter l_φ in merilo ekvatorja s iz Enačbe P. 1 s polinomom (kjer je $A_1 = s$). Polinom v Enačbi P. 15 vsebuje parametre s, k, π, in d_φ. Integracija teh konstant v polinome je bila narejena z enim samim namenom, in sicer zato, da se zmanjša število množenj in s tem omogočiti hitrejši preračun. Enačba P. 14 tako vsebuje člene s sodimi potencami geografske širine, brez šeste in osme, le enkrat pomnožene z geografsko dolžino. V Enačbi P. 15 so le členi lihe potence geografske širine, brez pete, in brez geografske dolžine.

$$X = R \cdot \lambda \cdot (A_1 + A_2\varphi^2 + A_3\varphi^4 + A_4\varphi^{10} + A_5\varphi^{12})$$ (Enačba P. 14)

$$Y = R \cdot (B_1\varphi + B_2\varphi^3 + B_3\varphi^7 + B_4\varphi^9 + B_5\varphi^{11})$$ (Enačba P. 15)

kjer so:

X in Y projekcijske koordinate,

φ in λ geografska širina in dolžina, podani v radianih,

R je polmer zemlje,

A_1 do A_5 in B_1 do B_5 so koeficienti, podani v Tabeli P. 3.

Tabela P. 3: Koeficienti polinomski enačb za Naravno Zemljino projekcijo.

Koeficienti za X		Koeficienti za Y	
A_1	0,870700	B_1	1,007226
A_2	-0,131979	B_2	0,015085
A_3	-0,013791	B_3	-0,044475
A_4	0,003971	B_4	0,028874
A_5	-0,001529	B_5	-0,005916

P.5.2 Inverzna projekcija

Inverzno projekcijo sestavljajo naslednji štirje koraki:

(1) Izračun začetne vrednosti geografske širine: $\varphi_0 = Y \cdot R^{-1}$

(2) S tangentno metodo sledi izračun geografske širine φ: $\varphi_{n+1} = \varphi_n - F(\varphi_n) \cdot (F'(\varphi_n))^{-1}$, kjer je $F(\varphi_n) = B_1\varphi_n + B_2{\varphi_n}^3 + B_3{\varphi_n}^7 + B_4{\varphi_n}^9 + B_5{\varphi_n}^{11} - Y \cdot R^{-1} = 0$, $F'(\varphi_n)$ njen odvod in $n = 0, 1, 2, \dots, m$. V m-tem koraku se iterativni proces ustavi, ko je izpolnjen pogoj $|\varphi_{m+1} - \varphi_m| < \varepsilon$, kjer je ε zadovoljivo majhno pozitivno število.

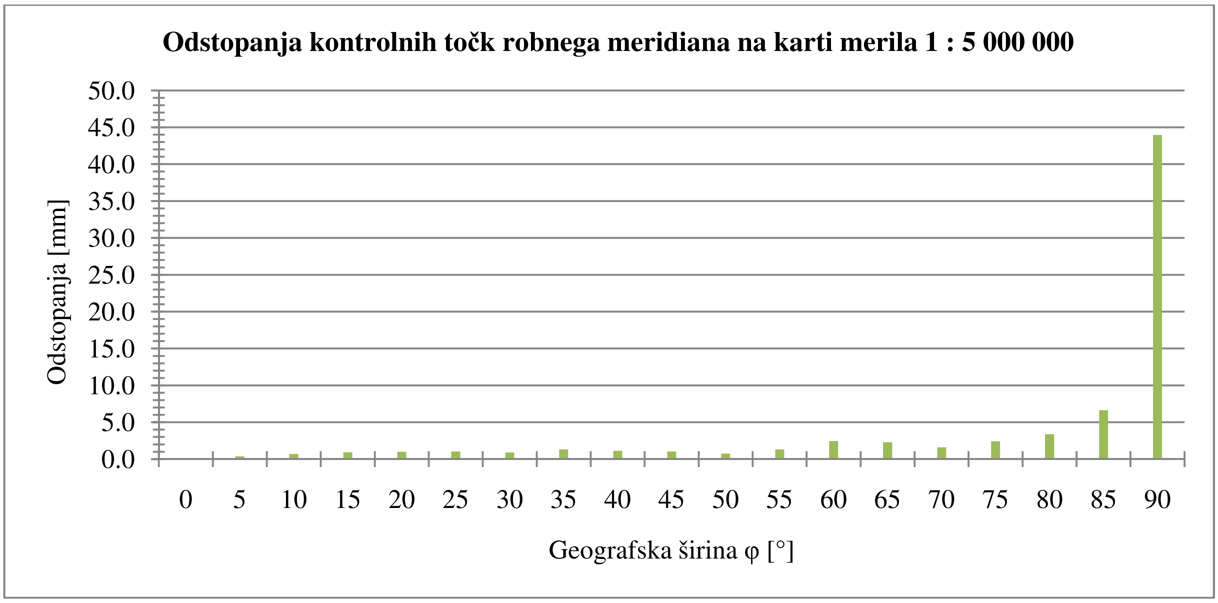
(3) Končna vrednost geografske širine: $\varphi = \varphi_{m+1}$

(4) Izračun geografske dolžine: $\lambda = X \cdot R^{-1} \cdot (A_1 + A_2\varphi^2 + A_3\varphi^4 + A_4\varphi^{10} + A_5\varphi^{12})^{-1}$

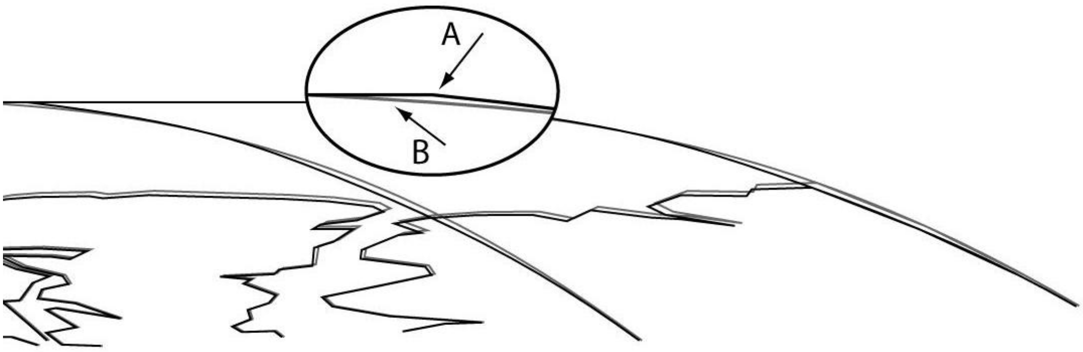
Tangentna metoda je pri tem uporabljena samo enkrat, in sicer v drugem koraku za rešitev Enačbe P. 15. Metoda konvergira hitro, v povprečju so potrebne manj kot 4 iteracije.

P.5.3 Primerjava izvirne in polinomske Naravne Zemljine projekcije

Primerjava obeh projekcij je bila izvedena na projekcijski mreži v merilu 1 : 5.000.000, kjer je širina karte 3,489 m in višina 1,814 m. Slika P. 4 predstavlja absolutne razlike 19 kontrolnih točk na robnem meridianu, $\lambda = 180°$, s katerimi je bila originalna projekcija tudi podana. Graf prikazuje večja odstopanja pri večjih vrednostih geografske širine, še posebej pri polu, kjer razlika znaša skoraj 45 mm. Razlog za to spremembo je seveda izboljšava zakrivljenih robov. Na Sliki P. 5 lahko vidimo, kako se je ta zakrivljenost spremenila in kakšne razlike so prisotne med mrežama obeh projekcij. Nekoliko večje odstopanje je pri 60. in 65. stopinjah, v ostalih točkah pa so vrednosti okoli 1 mm.



Slika P. 4: Odstopanja kontrolnih točk robnega meridiana na karti merila 1 : 5.000.000.



Slika P. 5: Izvirna (A) in izboljšana (B) Naravna Zemljina projekcija. Puščice nakazujejo spremembo gladkosti ob stičišču pola in robnega meridiana.

P.6 ZAKLJUČEK

Glavna naloga diplomske naloge je bila določitev analitične enačbe za Naravno Zemljino projekcijo. Za dosego cilja je bila uporabljena aproksimacijska metoda s polinomskimi enačbami. Oblikovanje enačb je temeljilo na osnovnih lastnostih projekcije in iskanju minimalnih absolutnih razlik med izvorno in izboljšano projekcijo. Predstavljeni polinomski enačbi, ki nadomeščata konstante in tabelarične vrednosti vsebujeta obvladljivo število parametrov in sta enostavni za preračun ter integracijo v računalniške programe. Ob pravilni faktorizaciji enačb zahtevata vsaka le sedem množenj. Projekcijo je mogoče tudi invertirati z uporabo tangentne metode, za katero v povprečju potrebujemo manj kot štiri korake.

Projekcija izboljšuje zakrivljenost stičišč in grafično podaja videz, da so sedaj le-ta zglajena. Zaradi tega lahko smatramo, da polinomske enačbe določajo izboljšano Naravno Zemljino kartografsko projekcijo. Avtorji članka "A polynomial equation of the Natural Earth projection" (Šavrič et al., sprejet za objavo) tako predlagajo uporabo polinomskih enačb (Enačbi P. 14 in 15 ter Tabela P. 3) kot pravo analitično enačbo za Naravno Zemljino projekcijo, ki bi nadomestila dosedanjo grafično določitev projekcije.

S tem sta izpolnjena tako glavni kakor tudi dodatni cilj te diplomske naloge. Do sedaj Naravno Zemljino kartografsko projekcijo že vključuje Java Map Projection Library (Dodatek, Apprendix C) in je seveda v seznamu projekcij, ki so vključene v aplikaciji Flex Projector. Objava polinomskih enačb pa bo omogočila širšo uporabo te kartografske projekcije tudi v drugih programskih orodjih. Prvi komercialni program, ki omogoča uporabo te kartografske projekcije, je tako že Global Mapper (Slika 14, stran 34). Ob izidu novih verzij programov GeoCart, Natural Scene Designer, PROJ.4, MAPublisher, Geographic Imager in drugih bodo ti prav tako vsebovali Naravno Zemljino kartografsko projekcijo.

Diplomska naloga predstavlja način določitve polinomske enačbe konkretno na primeru Naravne Zemljine kartografske projekcije, ki je bila grafično zasnovana v programu Flex Projector. Objava uporabljene metode v članku (Šavrič et al., sprejet za objavo) bo omogočila uporabo te tehnike tudi za druge projekcije, ki so bile zasnovane na enak način. Nerešena ostaja le umestitev te polinomsko aproksimacijske metode v Flex Projektor za katero koli projekcijo v tem programu.

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9 APPENDICES

APPENDIX A: Matlab code for the final determination of the polynimail coefficients

APPENDIX B: The code of Matlab function for the inverse transformation

APPENDIX C: Java Code for implementing the Natural Earth Projection

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APPENDIX A:
MATLAB CODE FOR THE FINAL DETERMINATION OF THE POLYNOMAIL COEFFICIENTS

This appendix provides the Matlab code, which was used for the final calculation of the polynomial coefficients. The Matlab code takes into consideration all constraints to preserve the dimensions of the graticule, and two additional measures to increase the smoothness of the rounded corners. Since the used functional model is linear, no special loop is required for this determination.

```
% POLYNOMIAL APPROXIMATION FOR NATURAL EARTH PROJECTION
%
% Copyright [2011] [Bojan Savric]
%
% Licensed under the Apache License, Version 2.0 (the "License");
% you may not use this file except in compliance with the License.
% You may obtain a copy of the License at
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%     http://www.apache.org/licenses/LICENSE-2.0
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% Unless required by applicable law or agreed to in writing, software
% distributed under the License is distributed on an "AS IS" BASIS,
% WITHOUT WARRANTIES OR CONDITIONS OF ANY KIND, either express or implied.
% See the License for the specific language governing permissions and
% limitations under the License.
%
% author: Bojan Savric, in collaboration with Tom Patterson, US National
% Park Service, and Bernhard Jenny, Institute of Cartography, ETH Zurich
%
%
% lengthP*s = A1 + A2*latitude^2 + A3*latitude^4 + A4*latitude^10 +
%             + A5*latitude^12
% distanceP*s*k*Pi = B1*latitude + B2*latitude^3 + B3*latitude^7 +
%                    + B4*latitude^9 + B5*latitude^11

clear all;
clc
format long;


%STARTING DATA
s = 0.8707; %scale factor
k = 0.52; %height-to-width ratio

%on the whole defined area there are 37 points with known length and
%distance of parallels:
latitude = [-90, -85, -80, -75, -70, -65, -60, -55, -50, -45, -40, -35,...
    -30, -25, -20, -15, -10, -5, 0, 5, 10, 15, 20, 25, 30, 35, 40, 45,...
    50, 55, 60, 65, 70, 75, 80, 85, 90]'*pi()/180; %unit is rad
lengthP = [0.55, 0.627, 0.6754, 0.716, 0.7525, 0.7874, 0.8196, 0.8492,...
    0.8763, 0.9006, 0.9222, 0.9409, 0.957, 0.9703, 0.9811, 0.9894,...
    0.9953, 0.9988, 1.0, 0.9988, 0.9953, 0.9894, 0.9811, 0.9703, 0.957,...
    0.9409, 0.9222, 0.9006, 0.8763, 0.8492, 0.8196, 0.7874, 0.7525,...
    0.716, 0.6754, 0.627, 0.55]'*s;
```

```
distanceP = [-1.0, -0.9761, -0.9394, -0.8936, -0.8435, -0.7903, -0.7346,...
    -0.6769, -0.6176, -0.5571, -0.4958, -0.434, -0.372, -0.31, -0.248,...
    -0.186, -0.124, -0.062, 0.0, 0.062, 0.124, 0.186, 0.248, 0.31,...
    0.372, 0.434, 0.4958, 0.5571, 0.6176, 0.6769, 0.7346, 0.7903,...
    0.8435, 0.8936, 0.9394, 0.9761, 1.0]'*s*k*pi();

n = length(latitude);


%LEAST SQUARES METHOD
%matrix of coefficients for both polynomial approximations
% A_1 is for the length and A_2 is for the distance of parallels
for i=1:n
    A_1(i,:)= [1, latitude(i)^2, latitude(i)^4, latitude(i)^10,...
        latitude(i)^12];
    A_2(i,:)= [latitude(i), latitude(i)^3, latitude(i)^7, latitude(i)^9,...
        latitude(i)^11];
end


%LSM for the length of parallels
X0_1 = inv(A_1'*A_1)*A_1'*lengthP; %adjustment of indirect observations
d1 = length(X0_1);

%additional constraint - adding functionally dependence in LSM:
%(1) free coefficient A1 must be equal 1.0*s
C_1(1,d1)=0; C_1(1,1)=1; g1=lengthP(19);

M_1 = C_1*inv(A_1'*A_1)*C_1';
dX_1 = inv(A_1'*A_1)*C_1'*inv(M_1)*(g1-C_1*X0_1);

%unknowns with constraint included
X_1 = X0_1 + dX_1;
v_1 = A_1*X_1 - lengthP;
lengthP_1 = lengthP + v_1;
RMS(1) = sqrt(v_1'*v_1/(n - d1 + 1));


%LSM for the distance of parallels
X0_2 = inv(A_2'*A_2)*A_2'*distanceP; %adjustment of indirect observations
d2 = length(X0_2);

%additional constraints - adding functionally dependence in LSM:
%(2) the distance of parallels at 90° must be equal 1
%(3) the slope of this curve must be 7° at the latitude 90°
C_2(1,:)=[pi()/2, (pi()/2)^3, (pi()/2)^7, (pi()/2)^9, (pi()/2)^11];
C_2(2,:)=[1, 3*(pi()/2)^2, 7*(pi()/2)^6, 9*(pi()/2)^8, 11*(pi()/2)^10];
g2=[distanceP(n), tan(7*pi()/180)]';

M_2 = C_2*inv(A_2'*A_2)*C_2';
dX_2 = inv(A_2'*A_2)*C_2'*inv(M_2)*(g2-C_2*X0_2);

%unknowns with constraints included
X_2 = X0_2 + dX_2;
v_2 = A_2*X_2 - distanceP;
distanceP_2 = distanceP + v_2;
RMS(2) = sqrt(v_2'*v_2/(n - d2 + 2));
```

```
%PLOTING THE RESULTS OF THE POLYNOMIAL APPROXIMATIONS
fprintf('\nPOLYNOMIAL APPROXIMATION FOR NATURAL EARTH PROJECTION');
fprintf('\n\nlengthP * s =\n');
fprintf('= A1 + A2*latitude^2 + A3*latitude^4 + A4*latitude^10 +
A5*latitude^12\n');
for j=1:d1
    fprintf('A%d -> %9.6f\n', j, X_1(j));
end
fprintf('\ndistanceP * s*k*Pi =\n');
fprintf('= B1*latitude + B2*latitude^3 + B3*latitude^7 + B4*latitude^9 +
B5*latitude^11\n');
for j=1:d2
    fprintf('B%d -> %9.6f\n', j, X_2(j));
end
```

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APPENDIX B:

THE CODE OF MATLAB FUNCTION FOR THE INVERSE TRANSFORMATION

This appendix provides the code of a Matlab function, which presents the inverse procedure for the polynomial Natural Earth projection. The Newton-Raphson method is used, and it requires iterations with while loop.

```
function [phi,lam]=NEP_xy2philam(X_p,Y_p,R)
%function [phi,lam]=NEP_xy2philam(X_p,Y_p,R) is the inverse transformation
%for the Natural Earth projection.
%
%Cartesian X/Y coordinates and radius R of the generating globe are
%required in meters. The output consist of spherical coordinates phi and
%lam in radians units.
%
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%
% author: Bojan Savric, in collaboration with Tom Patterson, US National
% Park Service, and Bernhard Jenny, Institute of Cartography, ETH Zurich


%DETERMINATION OF INITIAL GUESSES
Y_c = Y_p/R;
n = length(Y_c); %counting the numbers of points


%DETERMINATION OF LATITUDE
for i=1:n
    x = Y_c(i);
    %first approximation of solution
    xn = x - ((1.007226.*x + 0.015085.*x.^3 - 0.044475.*x.^7 +...
        0.028874.*x.^9 - 0.005916.*x.^11)-Y_c(i))/(1.007226 +...
        0.045255.*x.^2 - 0.311325.*x.^6 + 0.259866.*x.^8 -
0.065076.*x.^10);
    j=1;

    %Newton-Raphson loop
    while(abs(xn-x)>1e-13)
        x = xn;
        xn = x - ((1.007226.*x + 0.015085.*x.^3 - 0.044475.*x.^7 +...
            0.028874.*x.^9 - 0.005916.*x.^11)-Y_c(i))/(1.007226 +...
            0.045255.*x.^2 - 0.311325.*x.^6 + 0.259866.*x.^8 -...
```

```
            0.065076.*x.^10);
        j=j+1;

        if j==50 %exit in case of unending loop
           fprintf('ERROR!!! example: %d\n', i);
           break;
        end
    end

    %final latitude value, and number of iteration steps
    phi(i,1)=xn;
    Js(i)=j;
end


%DETERMINATION OF LONGITUDE
one(1:n,1)= 0.8707;
L_phi = one - 0.131979.*phi.^2 - 0.013791.*phi.^4 + 0.003971.*phi.^10 -...
    0.001529.*phi.^12;
lam = X_p./(R.*L_phi);


%PLOTING THE RESULTS OF INVERSE TRANSFORMATION
fprintf('\nTHE RESULTS OF INVERSE TRANSFORMATION FOR THE NATURAL EARTH
PROJECTION\n');
fprintf('\nThe numbers of iteration steps:\n');
for j=1:n
    fprintf('point: %d -> %3d\n', j, Js(j));
end

fprintf('\nThe spherical coordinates of the Natural Earth projection:\n');
fprintf('         phi [rad]   lam [rad]\n');
for j=1:n
    fprintf('%3d: -> %10.6f  %10.6f\n', j, phi(j), lam(j));
end
```

APPENDIX C:

JAVA CODE FOR IMPLEMENTING THE NATURAL EARTH PROJECTION

This appendix provides the Java code implementing the final polynomial equation of the Natural Earth projection in Flex Projector and in Java Map Projection Library (Jenny, B. 2011). The code contains forward and inverse projections.

```
/*
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 */
package com.jhlabs.map.proj;

import java.awt.geom.*;

/**
 * The Natural Earth projection was designed by Tom Patterson, US National
Park
 * Service, in 2007, using Flex Projector. The shape of the original
projection
 * was defined at every 5 degrees and piece-wise cubic spline interpolation
was
 * used to compute the complete graticule.
 * The code here uses polynomial functions instead of cubic splines and
 * is therefore much simpler to program. The polynomial approximation was
 * developed by Bojan Savric, in collaboration with Tom Patterson and
Bernhard
 * Jenny, Institute of Cartography, ETH Zurich. It slightly deviates from
 * Patterson's original projection by adding additional curvature to
meridians
 * where they meet the horizontal pole line. This improvement is by
intention
 * and designed in collaboration with Tom Patterson.
 *
 * @author Bojan Savric
 */
public class NaturalEarthProjection extends PseudoCylindricalProjection {

    private static final double A0 = 0.8707;
    private static final double A1 = -0.131979;
    private static final double A2 = -0.013791;
    private static final double A3 = 0.003971;
```

```
    private static final double A4 = -0.001529;
    private static final double B0 = 1.007226;
    private static final double B1 = 0.015085;
    private static final double B2 = -0.044475;
    private static final double B3 = 0.028874;
    private static final double B4 = -0.005916;
    private static final double C0 = B0;
    private static final double C1 = 3 * B1;
    private static final double C2 = 7 * B2;
    private static final double C3 = 9 * B3;
    private static final double C4 = 11 * B4;
    private static final double EPS = 1e-11;
    private static final double MAX_Y = 0.8707 * 0.52 * Math.PI;

    @Override
    public Point2D.Double project(double lplam, double lpphi,
Point2D.Double out) {

        double phi2 = lpphi * lpphi;
        double phi4 = phi2 * phi2;

        out.x = lplam * (A0 + phi2 * (A1 + phi2 * (A2 + phi4 * phi2 * (A3 +
phi2 * A4))));
        out.y = lpphi * (B0 + phi2 * (B1 + phi4 * (B2 + B3 * phi2 + B4 *
phi4)));

        return out;
    }

    @Override
    public boolean hasInverse() {
        return true;
    }

    @Override
    public Point2D.Double projectInverse(double x, double y, Point2D.Double
lp) {

        // make sure y is inside valid range
        if (y > MAX_Y) {
            y = MAX_Y;
        } else if (y < -MAX_Y) {
            y = -MAX_Y;
        }

        // latitude
        double yc = y;
        double tol;
        for (;;) { // Newton-Raphson
            double y2 = yc * yc;
            double y4 = y2 * y2;
            double f = (yc * (B0 + y2 * (B1 + y4 * (B2 + B3 * y2 + B4 *
y4)))) - y;
            double fder = C0 + y2 * (C1 + y4 * (C2 + C3 * y2 + C4 * y4));
            yc -= tol = f / fder;
            if (Math.abs(tol) < EPS) {
                break;
            }
        }
        lp.y = yc;
```

```
        // longitude
        double y2 = yc * yc;
        double phi = A0 + y2 * (A1 + y2 * (A2 + y2 * y2 * y2 * (A3 + y2 *
A4)));
        lp.x = x / phi;

        return lp;
    }

    @Override
    public String toString() {
        return "Natural Earth";
    }
}
```